RAYLEIGH-TAYLOR INSTABILITY WITH HEAT TRANSFER

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Presented to
The Academic Faculty

By

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RAYLEIGH-TAYLOR INSTABILITY WITH HEAT TRANSFER

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� **Author**: I request that the attached errata sheet be added to my thesis. I have attached two copies prepared as prescribed by the current *Specifications for Thesis Preparation*.

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Errata One:

(a) In the Acknowledgement section, Page vi;

(b) After the ending of the acknowledgement;

(c) Add in a new paragraph:

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You miss one hundred percent of the shots you don’t take.

Michael Jordan

To my mother and father.

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SUMMARY

In this thesis, the Rayleigh-Taylor instability effect with heat transfer in the setting of the Navier-Stokes equations, given three-dimensional and incompressible fluids, is investigated. Under suitable initial and boundary conditions, the existence of the temperature variable in the weak form is established by the Galerkin Method; the uniqueness of the solution can be proved by energy estimates, and furthermore, if more conditions on the coefficients are imposed, it can be demonstrated that with the help of properties on bounded operators, the temperature belongs to some Hölder continuous class.

As demonstrated by [24], given RT instability driven initial conditions, the exponential growth effect can be shown for the density and the velocity even in the fully nonlinear setting. By the regularity result above, a positive minimum temperature result can be established. Therefore, with the help of the known profiles of the instability in the density and the velocity, the instability for the temperature in the sense of Hadamard can be proved.

CHAPTER 1

INTRODUCTION AND BACKGROUND

Navier-Stokes equations are employed to describe the motions of viscous fluids in physics, and they are named after Claude-Louis Navier and George Stokes. The set of equations are derived from following the three laws in physics, namely the conservation of mass, the conservation of momentum (or Newton's Second Law), and the conservation of energy. With the force of gravity as the only external force, the equations can be written as:

$$\begin{aligned} \rho_t + div(\rho\mathbf{u}) &= 0, \\ (\rho\mathbf{u})_t + div(\rho\mathbf{u} \otimes \mathbf{u}) &= -\nabla P + div\mathbb{S} - \rho\mathbf{g}, \\ (\rho e)_t + div(\rho e\mathbf{u}) &= -P div\mathbf{u} + div(\kappa\nabla\theta) + \mathbb{S} : \nabla\mathbf{u}. \end{aligned} \tag{1.1}$$

Here we use ρ as the density of the substance, **u** as the velocity vector, e as the specific internal energy, θ as the absolute temperature, and P as the pressure. And **g** is force of gravity, $\mathbb{S}$ is the stress tensor as in (1.9). In this thesis we treat μ and κ as constants. With the assist of the equations of states, we can regard the systems as consisting of unknown variables(namely, ρ, **u**, P, e, θ) and five equations. The studies of Navier-Stokes equations, or similar type equations which intend to capture the mathematical backgrounds of fluids can be traced back to centuries ago; while important results and analyses are present, there are several basic and important questions on large time behaviors remaining about these types of system. We will continue the introduction firstly on the field equations, and later our attention will turn primarily to the incompressible flow.

1.1 Field Equations

1.1.1 Continuity Equation

In fluid mechanics, the continuity equation is employed to describe the conserved behavior of some substances, and it is a locally stronger form of the general conservation law, and it is constructed mathematically by the balance laws. Here $\rho(x,t)$ is the density variable depending on time and the spacial variables, and we use Ω as the region in context. Then

$$Mass(\Omega) = \int_{\Omega} \rho(x,t)dx. \tag{1.2}$$

is the mass of the substance. Applying the balance law from the conservation of mass, which states that the total rate of the mass flowing into the system is equal to the rate of the mass loss added by the rate of the accumulation of the mass, we first have the integral form of the continuity equation:

$$\int_{\Omega} \rho(x,t_2) - \rho(x,t_1) + \int_{t_1}^{t_2} \int_{\partial\Omega} \rho\mathbf{u}\cdot\mathbf{n}dSdt = 0. \tag{1.3}$$

Consequently we have the differential form of the equation as:

$$\rho_t + div(\rho\mathbf{u}) = 0. \tag{1.4}$$

The first term in equation (1.4) is the accumulation (or loss) of the mass, whereas the second term represents the difference of the flow in and out. Another form of the continuity equation can be given in terms of the material derivative of the density:

$$\frac{D\rho}{Dt} + \rho div(\mathbf{u}) = 0, \tag{1.5}$$

where $\frac{D\rho}{Dt} = \rho_t + \mathbf{u}\cdot\nabla\rho$.

1.1.2 Momentum Equation

By Newton's second law of motion, considering the fluids we would get:

$$\rho \frac{D\mathbf{u}}{Dt} = f = f_{force} + f_{surface}, \tag{1.6}$$

f_{force} is the body force on the total mass of the fluid, thus $f_{force} = \rho\mathbf{f}$, and $f_{surface}$ is given as the combination of the pressure and viscous forces in Navier-Stokes equation, and can be represented by $f_{surface} = div\mathbb{T}$. Here $\mathbb{T}$ is the stress tensor that consists of a deviatoric part and a volumetric part:

$$\mathbb{T} = \mathbb{S} - P\mathbb{I}. \tag{1.7}$$

Here $\mathbb{I}$ is the identity matrix. Then the equation representing the linear momentum conservation is:

$$(\rho\mathbf{u})_t + div(\rho\mathbf{u} \otimes \mathbf{u}) = -\nabla P + div\mathbb{S} + \rho\mathbf{f}, \tag{1.8}$$

The stress tensor can be expressed as:

$$\mathbb{S} = \mu(\nabla\mathbf{u} + \nabla\mathbf{u}^T - \frac{2}{3}div\mathbf{u}\mathbb{I}) + \eta div\mathbf{u}\mathbb{I}. \tag{1.9}$$

Here μ and η are the shear viscosity coefficient and the bulk viscosity coefficient respectively. In some cases they are both treated as non-negative constants. Another way of expressing the stress tensor $\mathbb{S}$ can be shown as:

$$\mathbb{S} = 2\mu D(\mathbf{u}) + \lambda(div\mathbf{u})\mathbb{I}, \tag{1.10}$$

in which μ is the viscosity coefficient, and:

$$D(\mathbf{u}) = \frac{1}{2}(\nabla\mathbf{u} + \nabla\mathbf{u}^T), \tag{1.11}$$

and the coeffient

$$\lambda = -\frac{2}{3}\mu + \eta. \tag{1.12}$$

In this thesis we assume **g**, the force of gravity, as the only external force acting on the substance, and in three dimensional space, **g** = (0, 0, g)

1.1.3 Energy Equation

In the first law of thermal-dynamcis, the increased energy, E_t, is viewed as the combination of work done and heat added into the system:

$$dE_t = dQ + dW, \tag{1.13}$$

Q, W denote the heat and the work respectively. One type of the several expressions of the energy equation could be expressed by:

$$(\rho e)_t + div(\rho e\mathbf{u}) = -div\mathbf{q} + \Phi - Pdiv\mathbf{u}, \tag{1.14}$$

where the heat flux term **q** is governed by the Fourier's law $\mathbf{q} = -\kappa\nabla\theta$, with κ as the thermal conductivity coefficient. While assuming κ constant, we arrive at:

$$(\rho e)_t + div(\rho e\mathbf{u}) = \kappa\Delta\theta + \Phi - Pdiv\mathbf{u}. \tag{1.15}$$

1.1.4 The incompressible Navier-Stokes Equations

If we restrict the conditions of the fluids to be incompressible, i.e., the fluid cannot be compressed nor expanded, then mathematically that means the material derivative of the density $\rho(x,t)$ is zero:

$$\frac{D\rho}{Dt} = \rho_t + \nabla\rho \cdot \mathbf{u} = 0. \tag{1.16}$$

Then the continuity equation can be simplified to $div\mathbf{u} = 0$, or can be expressed as the material derivative of the density being zero. In other words, when we follow one particular moving fluid element in the vector flows, the change of the density is zero. Express the density as a function of time and spatial variables:

$$\rho = \rho(t, \mathbf{X(t)}), \tag{1.17}$$

in which the spatial variables $\mathbf{X}(t) = (x_1(t), x_2(t), x_3(t))$ in the three dimensional space. Since we are tracking the movement of a fixed element, then under the incompressible condition, by taking the material derivative and keeping in mind presence of the divergence free condition, we naturally get (1.16).

One useful application of (1.16) is if the initial density $\rho_0 \in C(\Omega)$, where Ω is a bounded region Ω, then:

$$\underset{x\in\Omega}{inf}\{\rho_0(x)\} \leq \rho(x) \leq \underset{x\in\Omega}{sup}\{\rho_0(x)\}. \tag{1.18}$$

Also, when expressing the stress tensor term in the momentum equation, for incompressible fluid the expression can be simplified to:

$$\mathbb{S} = 2\mu D(\mathbf{u}) = \mu(\nabla\mathbf{u} + \nabla\mathbf{u}^T), \tag{1.19}$$

thus the viscosity term can be simplified to:

$$div(\mathbb{S}) = \mu\nabla \cdot (\nabla\mathbf{u} + \nabla\mathbf{u}^T). \tag{1.20}$$

Then the original momentum equation in the incompressible flows can be presented as:

$$\rho\mathbf{u}_t + \rho(\mathbf{u} \cdot \nabla)\mathbf{u} = -\nabla P + \mu\Delta\mathbf{u} + \rho\mathbf{g}. \tag{1.21}$$

The expression $\rho(\mathbf{u} \cdot \nabla)\mathbf{u}$ is the convective nonlinear term in the momentum equation, and $\mu\Delta\mathbf{u}$ is the viscous term.

1.1.5 The equations of state and further assumptions

Before further discussing the energy equation, a general introduction of the equations of state should be in place. For the compressible flows, there are several equations of state that aim to capture the relationships between the state variables such as pressure, specific internal energy,volume, and temperature, as presented in various literatures [7], [29]. For instance, the pressure can be interpreted as a function of the density and the temperature:

$$P = P(\rho, \theta), \tag{1.22}$$

also $e = e(\rho, \theta)$. Internally they are governed by the Maxwell's relation:

$$\frac{\partial e(\rho, \theta)}{\partial \rho} = \frac{1}{\rho^2}(P(\rho, \theta) - \theta\frac{\partial P(\rho, \theta)}{\partial \theta}). \tag{1.23}$$

In the incompressible flow case, we assume the change of density to be negligible, and we use:

$$P = P(x, t) \tag{1.24}$$

as an approximation for the pressure, replacing the equations of state. For a fixed volume, $de = C_v d\theta$, C_v as the heat capacity coefficient, thus substituting the specific internal energy e, we can view the energy function as a function primarily of the temperature θ. Since in the thesis we are concerned with the existence and the exponential/instability behavior of the variables, after some adjustment to the constants we arrive at:

$$(\rho\theta)_t + div(\rho\theta\mathbf{u}) = \kappa\Delta\theta + \Phi - Pdiv\mathbf{u}. \tag{1.25}$$

Applying the incompressible condition, we further get:

$$\rho(\theta)_t + \rho div(\theta \mathbf{u}) = \kappa \Delta \theta + \Phi. \tag{1.26}$$

Due to the absence of the equations of state in the incompressible flows, the energy equation is decoupled from the continuity and the momentum equations in the sense that θ does not show up in the continuity and momentum equations. Thus, we can analyze the properties of the velocity and pressure first, and then apply their known patterns into the energy equation to find estimates for the temperature term. In the energy equation, Φ can be explicitly written as:

$$\Phi = \mathbb{S} : \nabla \mathbf{u}. \tag{1.27}$$

1.1.6 Symmetrized gradient and the Korn's Inequality

In incompressible flows, divergence of the velocity term in $\mathbb{S}$ vanished (1.9), and:

$$\mathbb{S} = 2\mu D\mathbf{u}. \tag{1.28}$$

Thus from (1.19), (1.27), we get:

$$\Phi = \frac{|\nabla \mathbf{u} + \nabla \mathbf{u}^T|^2}{2}. \tag{1.29}$$

By including and appending higher order nonlinear terms in the Navier-Stokes equations (for instance Burnett-type equations), we can acquire more accurate fluid motions. Navier-Stokes and the many Burnett equations can be derived fully from the Boltzmann Equation by Chapman-Enskog method [30]. If we view $\rho, \mathbf{u}, \theta$ as the primary variables of interest, then the Chapman-Enskog expansion at first order gives Navier-Stokes equations, whereas higher order expansions would result in Burnett Equations and super Burnett Equations.

For instance in two dimensional space, if

$$\mathbf{u} = (u_1(x,y), u_2(x,y)), \tag{1.30}$$

The symmetrized gradient term in (1.29) can be calculated as

$$\begin{aligned} |\nabla \mathbf{u} + \nabla \mathbf{u}^T|^2 &= 4(\frac{\partial u_1}{\partial x})^2 + 4(\frac{\partial u_2}{\partial y})^2 + 2(\frac{\partial u_1}{\partial y} + \frac{\partial u_2}{\partial x})^2 \\ &= 4(\frac{\partial u_1}{\partial x})^2 + 4(\frac{\partial u_2}{\partial y})^2 + 2(\frac{\partial u_1}{\partial y})^2 + 2(\frac{\partial u_2}{\partial x})^2 + 4\frac{\partial u_1}{\partial y} \cdot \frac{\partial u_2}{\partial x}. \end{aligned} \tag{1.31}$$

in 2D. When $\mathbf{u}$ vanishes on the boundary, employ some integration by parts and apply the boundary condition to get:

$$\begin{aligned} \int |\nabla \mathbf{u} + \nabla \mathbf{u}^T|^2 d\mathbf{x} - 2\int |\nabla \mathbf{u}|^2 d\mathbf{x} &= \int 2(\frac{\partial u_1}{\partial x})^2 + 2(\frac{\partial u_2}{\partial y})^2 + 4\frac{\partial u_1}{\partial y}\frac{\partial u_2}{\partial x} d\mathbf{x} \\ &= \int 2(\frac{\partial u_1}{\partial x})^2 + 2(\frac{\partial u_2}{\partial y})^2 + 4\frac{\partial u_1}{\partial x}\frac{\partial u_2}{\partial y} d\mathbf{x} \\ &= 2\int (div(\mathbf{u}))^2 d\mathbf{x}. \end{aligned} \tag{1.32}$$

The calculation in 3D would be similar. From here we can deduce the Korn's inequality:

$$\int |\nabla \mathbf{u} + \nabla \mathbf{u}^T|^2 d\mathbf{x} \geq 2\int |\nabla \mathbf{u}|^2 d\mathbf{x}. \tag{1.33}$$

Also in the incompressible flow case, we would have the two sides equal. By similar calculations, the inequality and the equation would also hold in the three dimensional case.

1.1.7 Euler Equations

Euler Equations can be viewed as a particular case of the Navier-Stokes equations when the viscosity and the thermal conductivity are considered to be zero. Named after Leonhard Euler, it was among the first partial differential equations ever created. For the compressible flows, coupled with energy equation, the Euler equations are:

$$\begin{aligned}
&\rho_t + div(\rho\mathbf{u}) = 0, \\
&\rho\mathbf{u}_t + \rho\mathbf{u}\cdot\nabla\mathbf{u} = -\nabla P + \rho\mathbf{g}, \\
&e_t + u\cdot\nabla e + \frac{P}{\rho}div(\mathbf{u}) = 0.
\end{aligned} \tag{1.34}$$

It is useful to consider the Euler equations of the incompressible flows. When $div\mathbf{u} = 0$, the energy equation reads:

$$e_t + u\cdot\nabla e = \frac{De}{Dt} = 0. \tag{1.35}$$

Thus e is constant along the flows, and ρe, the internal energy E satisfies the continuity equation in the conservation form. Thus often we only consider the continuity and momentum equation for the incompressible, inviscid fluids:

$$\begin{aligned}
&div\mathbf{u} = 0, \\
&\rho\mathbf{u}_t + \rho\mathbf{u}\cdot\nabla\mathbf{u} + \nabla P = \rho\mathbf{g}.
\end{aligned} \tag{1.36}$$

1.2 The Rayleigh–Taylor instability

1.2.1 History of the Rayleigh-Taylor Instability

The Rayleigh-Taylor Instability, or RT instability for short, is the physical phenomenon that occurs when a heavier density fluid is placed on top of a lighter fluid. Due to the force of gravity, the lighter fluid will push the heavier fluid on top. In term of fluid mechanics, the two fluids seek to reduce their total potential energy in this process.

The physics applications within the field of Rayleigh-Taylor instability are wide-spread. Lord Rayleigh [35] began the theoretical study for such instabilities from the nineteenth century, when he published the paper on this subject, inspired by the then famous discussion of the cirrus clouds. After almost 70 years, this kind of instability picked up its popularity when G. T. Taylor and Lewis began their experimental works around 1950, confirming the exponential growth rate character introduced by Rayleigh. Since then, the scope of the RT instability involves not only the constant gravitational field, but also changing force fields with acceleration.

1.2.2 Two phased fluids instability

In Rayleigh's original paper [35], he considered two incompressible immiscible fluids in 2D, with the steady state:

$$\overline{\mathbf{u}} = (u_1(x,y), u_2(x,y)) = 0, \quad \overline{\rho} = \rho_0. \tag{1.37}$$

The line that separates the two fluids is given by $y = 0$, with the gravity $\mathbf{g} = (0, -g)$. In this setting, he considered the linearized system: if k is used to denote the spatial wave number, **A** as the Atwood number. Here ρ_h is the heavy fluid's density, ρ_l is the lower fluid's density. Then the Atwood number is given by:

$$A = \frac{\rho_h - \rho_l}{\rho_h + \rho_l}. \tag{1.38}$$

He proved that the perturbation growth takes the exponential growth rate Λ, with:

$$\Lambda = \sqrt{Agk}. \tag{1.39}$$

It was quite remarkable for his time, Rayleigh had proposed such fundamental results to this topic. More importantly, he provided a breaking ground to one of the systematic approaches to tackle the instabilities for equations in non-linear settings.

In terms of the two phased fluids instability, one of the formulations introduced by Jan Pruss and Gieri Simonett [34] (they implemented Navier Stokes equations with similar settings), and discussed by Yan Guo and Ian Tice [18] in a 3D, compressible Euler equations is:

$$\begin{aligned} \partial_t \rho_\pm + div(\rho_\pm \mathbf{u}_\pm) = 0, \\ \rho_\pm(\partial_t \mathbf{u}_\pm + \mathbf{u}_\pm \cdot \nabla \mathbf{u}_\pm) + \nabla P_\pm(\rho_\pm)) = -\rho_\pm \mathbf{g}. \end{aligned} \tag{1.40}$$

In the papers they follow the general procedures as mentioned above. We will be focusing on the incompressible RT instabilities with a single continuous density profile fluid henceforth. The general strategy to demonstrate non-linear instability can be briefly stated as the following:

· Find suitable initial and boundary conditions, and a steady state for the original equations;

· Perturb around the steady state, and construct the linearized perturbed equations;

· For the unknowns in the linear equations such as ρ and u, set up a exponential growing ansatz in time, and find the sharp exponential growth rate usually by an application of the Gronwall's inequality;

· Energy estimates to investigate the general nonlinear equations based on the linear stability.

1.2.3 Incompressible instability with unbounded domain

One classical model developed in recent years on RT instability is the model proposed by Hyung and Yan in their 2003 research paper [21]. Here the incompressible Euler equations in 2D are considered:

$$\begin{aligned} &\rho_t + \mathbf{u} \cdot \nabla \rho = 0, \\ &\rho \mathbf{u}_t + \rho \mathbf{u} \cdot \nabla \mathbf{u} = -\nabla P + \rho \mathbf{g}, \\ &div \mathbf{u} = 0. \end{aligned} \tag{1.41}$$

with the domain:

$$D = \{-\infty < x < \infty, 0 \leq y \leq 2\pi\}. \tag{1.42}$$

The gravity will impose a force from left to right: $\mathbf{g} = (g, 0)$. Also the Neumann boundary condition is proposed, $\frac{\partial \mathbf{u}}{\partial \mathbf{n}} = 0$, where $\mathbf{n}$ is the outward unit normal. Instead of considering two different types of fluids, a varied-density fluid is in the equation, with a steady density profile $\overline{\rho}$ that can result in the instability under perturbation:

$$\lim_{x \to -\infty} \overline{\rho}(x) = \overline{\rho}^h, \quad \lim_{x \to \infty} \overline{\rho}(x) = \overline{\rho}^l, \tag{1.43}$$

The suitable steady state for the velocity $\overline{\mathbf{u}}$ is zero, and the steady pressure profile should satisfy $\nabla \overline{P} = \overline{\rho}\mathbf{g}$. When the perturbation is described as:

$$\sigma = \rho - \overline{\rho}, \mathbf{v} = \mathbf{u} - \mathbf{0}, p = P - \overline{P}, \tag{1.44}$$

the perturbed equations can be shown as:

$$\begin{aligned} \sigma_t + \mathbf{v} \cdot \nabla(\sigma + \overline{\rho}) &= 0, \\ (\sigma + \overline{\rho})\mathbf{v}_t + (\sigma + \overline{\rho})\mathbf{v} \cdot \nabla \mathbf{v} + \nabla p &= \sigma \mathbf{g}, \\ div \mathbf{v} &= 0. \end{aligned} \tag{1.45}$$

The final goal here is to examine if any instability profile exists for this fully nonlinear system, or in other words we are trying to prove:

The steady state profile described above is nonlinear unstable in the sense of Hadamard.

In order to do so, we construct the linearized system to (1.34):

$$\begin{aligned} \sigma_t + \overline{\rho}_x \cdot v_1 &= 0, \\ \overline{\rho}\mathbf{v}_t + \nabla p &= \sigma \mathbf{g}, \\ div \mathbf{v} &= 0. \end{aligned} \tag{1.46}$$

We then set up the normal growth mode of the linearized equations by the Fourier transform. For instance, for the linearized density profile σ, it takes the form:

$$\sigma(t,x,y) = \tilde{\sigma}(x)cos(ky)exp(\lambda_k t). \tag{1.47}$$

Here k is the wave number, and λ_k is the exponential growth rate defined by:

$$\lambda_k^2 = \sup_{u \in H^1(\mathbb{R})} \frac{\int -g\overline{\rho}_x u^2 dx}{\int \overline{\rho}[\frac{u_k^2}{k^2} + u^2]dx}. \tag{1.48}$$

One key point here is by the method the steady state density is defined, we would have at some point x, $\rho_x < 0$, thus λ_k is well defined. To regulate the growing coefficients for different wave numbers,

$$\Lambda^2 = \sup_{u \in L^2(\mathbb{R}\times\mathbb{T})} \frac{\int -g\overline{\rho}_x u^2 dxdy}{\int \overline{\rho} u^2 dxdy} \tag{1.49}$$

is introduced, and it can be proved that $\lambda_k \leq \lambda_{k+1}$, and λ_k tends to Λ from below. We then can prove the solutions to the linearized equations are controlled by Λ, then after estimations of the differences between the linearized equations and the perturbed equations, we can prove the instability for the nonlinear system.

1.2.4 Incompressible instability with bounded domain

Consider the 3D incompressible Navier-Stokes equation in a bounded domain Ω [24]:

$$\begin{aligned} \rho_t + \mathbf{u}\cdot\nabla\rho &= 0, \\ \rho\mathbf{u}_t + \rho\mathbf{u}\cdot\nabla\mathbf{u} &= -\nabla P + \mu\Delta\mathbf{u} - \rho\mathbf{g}, \\ div\mathbf{u} &= 0, \end{aligned} \tag{1.50}$$

with a steady state:

$$\bar{\mathbf{u}} = 0, \nabla \overline{P} = -\overline{\rho}\mathbf{g}. \tag{1.51}$$

The boundary condition is given as $\mathbf{u} = 0$ on $\partial\Omega$. The gravity is given as $\mathbf{g} = (0, 0, g)$, acting on the fluid in the negative x_3 direction. Thus the steady state density $\overline{\rho}$ only depends on the third component. With the perturbation around the steady state:

$$\sigma = \rho - \overline{\rho}, \mathbf{v} = \mathbf{u} - \mathbf{0}, p = P - \overline{P}, \tag{1.52}$$

We can find the perturbed function, and then after doing linearization:

$$\begin{aligned} \sigma_t + \overline{\rho}' u_3 &= 0, \\ \overline{\rho}\mathbf{v}_t + \nabla p &= \mu \Delta \mathbf{v} - \sigma \mathbf{g}, \\ div \mathbf{v} &= 0. \end{aligned} \tag{1.53}$$

where $\overline{\rho}'$ denotes $\frac{\partial \overline{\rho}}{\partial x_3}$. For the RT instability case, we consider the density profile satisfying:

$$inf\{\overline{\rho}(x)\} > 0, \quad inf\{\overline{\rho}'(x)\} > 0. \tag{1.54}$$

Motivated by the methods above, we are still looking for a first the linear exponential growing mode for the variables:

$$(\sigma, \mathbf{v}, p) = e^{\lambda t}(\tilde{\sigma}, \tilde{\mathbf{v}}, \tilde{p}), \tag{1.55}$$

for some growth rate λ. However, one of the main difficulties here is with the bounded domain, we cannot adopt the Fourier Transform with adjustable wave numbers. After plugging (1.55) into the linearized equations (1.53), and substitute $\tilde{\sigma}$ for $\tilde{\mathbf{v}}$ we get:

$$\begin{aligned} \Lambda^2 \overline{\rho} \tilde{\mathbf{v}} + \Lambda \nabla \tilde{p} &= \Lambda \mu \Delta \tilde{\mathbf{v}} + \mathbf{g} \overline{p}' \tilde{v}_3, \\ div \tilde{v} &= 0, \end{aligned} \tag{1.56}$$

with a compatible boundary condition $\tilde{\mathbf{v}}|_{\partial\Omega} = 0$. To circumvent the problem caused by the bounded domain, we employ a variational method introduced by Guo and Tice; we modify the system into:

$$\begin{aligned} \Lambda^2 \overline{\rho} \tilde{\mathbf{v}} + \Lambda \nabla \tilde{p} &= s \mu \Delta \tilde{\mathbf{v}} + \mathbf{g} \overline{\rho}' \tilde{v}_3, \\ div \tilde{v} &= 0. \end{aligned} \tag{1.57}$$

The above first equation can be transformed into

$$\Lambda^2 \int \overline{\rho} \tilde{\mathbf{v}}^2 dx = g \int \overline{\rho}' \tilde{v}_3^2 - s\mu \int |\nabla \tilde{\mathbf{v}}|^2 dx \tag{1.58}$$

by multiplying $\tilde{\sigma}$ on both sides and do integration. Hence we define an energy function

$$E(\tilde{\mathbf{v}}) := g \int \overline{\rho}' \tilde{v}_3^2 - s\mu \int |\nabla \tilde{\mathbf{v}}|^2 dx, \tag{1.59}$$

and an admissible set

$$A := \{\tilde{\mathbf{v}} \in H_0^1(\Omega), div \tilde{\mathbf{v}} = 0 | \int \overline{\rho} \tilde{\mathbf{v}}^2 dx = 1\}. \tag{1.60}$$

Thus we are looking for the growth rate Λ by finding the sup the energy functional within the admissible set. Also notice here the energy function is also dependent upon the choice of s: $E(\tilde{\mathbf{v}}) = E(\tilde{\mathbf{v}}, s)$. Define $\alpha(s) = \underset{\tilde{v} \in A}{sup} E(\tilde{\mathbf{v}}, s)$. By analyzing the characters of $\alpha(s)$, we get $\alpha(s)$ would be positive on an interval $(0, a)$; moreover,

$$\underset{s \to 0}{lim} \alpha(s) > 0, \quad \underset{s \to a}{lim} \alpha(s) = 0. \tag{1.61}$$

Then by a standard fix point argument, there exists Λ such that $\Lambda^2 = \underset{\tilde{v}\in A}{sup} E(\tilde{\mathbf{v}}, \Lambda)$. Then after showing that the maximizer can be attained, we have gained the exponential growth profile for the linearized equations. Then after more careful analyses we further prove that Λ is the sharp growth rate for $\sigma, \mathbf{v}$ under some suitable norm, and then by energy estimates with the help of the exponential growth linear profile, we get the nonlinear instability in the sense of Hadamard.

CHAPTER 2

THE INSTABILITIES OF THE TEMPERATURE VARIABLE

2.1 Formulation of the problem

Consider the following incompressible Naiver-Stokes Burnett type equations in 3D:

$$\begin{aligned} \rho_t + \mathbf{u} \cdot \nabla \rho &= 0, \\ \rho \mathbf{u}_t + \rho \mathbf{u} \cdot \nabla \mathbf{u} + \nabla P &= \mu \Delta \mathbf{u} - \rho \mathbf{g}, \\ \rho \theta_t + \rho \mathbf{u} \cdot \nabla \theta - \kappa \Delta \theta &= \frac{\mu}{2} |\nabla \mathbf{u} + \nabla \mathbf{u}^T|^2, \\ div \mathbf{u} &= 0. \end{aligned} \tag{2.1}$$

Here $\rho, \mathbf{u}, \theta$ denotes the density, velocity, and the absolute temperature variables. Since the third equation is decoupled from the previous continuity and momentum equations, we can study the energy equation separately. Here it is natural to impose one initial condition driven by the third law of thermal-dynamics::

$$\theta_0 > C \tag{2.2}$$

in a bounded region Ω. Define a steady state $\overline{\theta} \equiv C$, then the perturbed temperature $\tilde{\theta} = \theta - \overline{\theta}$ satisfies:

$$\tilde{\theta}_0 > 0. \tag{2.3}$$

We look for the instability temperature profile around the steady state in the following equation:

$$\rho \tilde{\theta}_t + \rho \mathbf{u} \cdot \nabla \tilde{\theta} - \kappa \Delta \tilde{\theta} = \frac{\mu}{2} |\nabla \mathbf{u} + \nabla \mathbf{u}^T|^2. \tag{2.4}$$

Since the equation remains unaltered, and we cannot guarantee the perturbed temperature variable $\tilde{\theta}$ to be positive, for the calculations below we will stick with the unperturbed equation. To impose one suitable boundary condition, we assume that θ is smooth in the following calculations; one integration of (2.4), and we get:

$$\int \rho\theta_t dx + \int \rho\mathbf{u} \cdot \nabla\theta dx = \kappa \int \Delta\theta dx + \frac{\mu}{2} \int |\nabla\mathbf{u} + \nabla\mathbf{u}^T|^2 dx \tag{2.5}$$

Since $u = 0$ on $\partial\Omega$, $\int div(\rho\mathbf{u}\theta)dx = \int \rho\mathbf{u}\theta \cdot \mathbf{n}dS = 0$, thus refering to the equation of continuity,

$$\begin{aligned} \frac{d}{dt}\int (\rho\theta)dx &= \frac{d}{dt}\int (\rho\theta)dx + \int div(\rho\mathbf{u}\theta)dx \\ &= \int \rho_t\theta dx + \int \rho\theta_t dx + \int \rho\mathbf{u} \cdot \nabla\theta dx + \int div(\rho\mathbf{u})\theta dx \\ &= \int \rho\theta_t dx + \int \rho\mathbf{u} \cdot \nabla\theta dx + \int (\rho_t + div(\rho\mathbf{u}))\theta dx \\ &= \int \rho\theta_t dx + \int \rho\mathbf{u} \cdot \nabla\theta dx. \end{aligned} \tag{2.6}$$

Also for the diffusion term on the right in (2.5),

$$\kappa \int \Delta\theta dx = \kappa \int \nabla\theta \cdot \mathbf{n}dS \tag{2.7}$$

Therefore, (2.5) can be viewed as

$$\frac{d}{dt}\int (\rho\theta)dx = \kappa \int \nabla\theta \cdot \mathbf{n}dS + \frac{\mu}{2} \int |\nabla\mathbf{u} + \nabla\mathbf{u}^T|^2 dx \tag{2.8}$$

From here we can see it is natural to consider the L^1 norm of the temperature when monitoring the system's instability behavior. Also since we will assume the initial density is positively bounded, the density inside the integral will not be an issue, thus as a first step, we impose the Neumann boundary condition for θ:

$$\frac{\partial \theta}{\partial \mathbf{n}} = 0 \quad on \quad [0,T] \times \partial\Omega. \tag{2.9}$$

Here T denotes the maximal time of the existence of the solution to the system, and the exact conditions of the system will be discussed shortly. Thus we have from (2.8)

$$\frac{d}{dt}\int (\rho\theta)dx = \frac{\mu}{2}\int |\nabla \mathbf{u} + \nabla \mathbf{u}^T|^2 dx \tag{2.10}$$

By the Korn's inequality/equality, recalling the divergence free condition, we arrive at

$$\frac{d}{dt}\int (\rho\theta)dx = \mu \int |\nabla \mathbf{u}|^2 dx \tag{2.11}$$

2.1.1 Conditions and estimates for ρ and $\mathbf{v}$

Before diving into the analyses of the energy equation, we need estimates and some detailed properties for the density and the velocity. The perturbed continuity equation and momentum equations are

$$\begin{aligned} \sigma_t + \mathbf{v}\cdot\nabla(\sigma+\overline{\rho}) &= 0, \\ (\sigma+\overline{\rho})\mathbf{v}_t + (\sigma+\overline{\rho})\mathbf{v}\cdot\nabla\mathbf{v} + \nabla p &= \mu\Delta\mathbf{v} - \sigma\mathbf{g}, \\ div\mathbf{v} &= 0. \end{aligned} \tag{2.12}$$

in which $\sigma, \mathbf{v}, p$ are the perturbed density, velocity and pressure:

$$\sigma = \rho - \overline{\rho}, \mathbf{v} = \mathbf{u} - \mathbf{0}, p = P - \overline{P}, \tag{2.13}$$

with the initial and boundary conditions:

$$(\sigma, \mathbf{v})|_{t=0} = (\sigma_0, \mathbf{v}_0), \mathbf{v}|_{\partial\Omega} = 0, div(\mathbf{v}_0) = 0. \tag{2.14}$$

Also we impose the RT instability profile requirements:

$$\overline{\rho} \in C^2,\ \inf_{x\in\Omega}\{\overline{\rho}(x) > 0\},\ \inf_{x\in\Omega}\{\partial_{x_3}\overline{\rho} > 0\}. \tag{2.15}$$

After linearization, the perturbed system is:

$$\begin{aligned} \sigma_t + \partial_{x_3}\overline{\rho} v_3 &= 0 \\ \overline{\rho}\mathbf{v}_t + \nabla p &= \mu\Delta\mathbf{v} - \sigma\mathbf{g} \\ div(\mathbf{v}) &= 0. \end{aligned} \tag{2.16}$$

in which v_3 denotes the third component of $\mathbf{v}$. The fully nonlinear instability is proved by F. Jiang and S. Jiang [24] in the following theorem:

Theorem 2.1.1 (By F. Jiang and S. Jiang) *Assume that the RT density profile $\overline{\rho}$ satisfies (2.15). Then the steady state of the system is unstable in the Hadamard sense: for positive constants Λ, m_0, there exists a positive constant ϵ, such that for any positive δ smaller than a constant δ_0, the initial data $(\sigma_0, \boldsymbol{v}_0) := \delta(\overline{\sigma}_0, \overline{\boldsymbol{v}_0})$ with $(\overline{\sigma}_0, \overline{\boldsymbol{v}_0}) \in H^2 \times H^2$ can generate a class of strong solutions $(\sigma, \boldsymbol{v}) \in C^2([0, T^{max}, H^1 \times H^2])$ to the perturbed equations (2.15) with an associated pressure $p \in H^1$, such that:*

$$\sup_{0\le t\le T^\delta}\{\inf\{\|\sigma(t)\|_{L^2}, \|(v_1, v_2)(t)\|_{L^2}, \|v_3(t)\|_{L^2}\}\} \ge \epsilon. \tag{2.17}$$

The escape time

$$T^\delta := \frac{1}{\Lambda}log\frac{2\epsilon}{m_0\delta} \in (0, T^{max}) \tag{2.18}$$

where T^{max} is the maximal time of the existence of the solution $(\sigma, \boldsymbol{v})$.

Here we provide the outline of the proof:

Step 1. For the linearized solution we construct an exponential growth mode:

$$\sigma(x) = \tilde{\sigma}e^{\Lambda t}, \mathbf{v} = \tilde{\mathbf{v}}e^{\Lambda t}, p(x) = \tilde{p}e^{\Lambda t}, \tag{2.19}$$

and by plugging into the linearized equation:

$$\begin{aligned}
\Lambda\tilde{\sigma} + \partial_{x_3}\overline{\rho}\tilde{\mathbf{v}}_3 = 0 \\
\Lambda\overline{\rho}\tilde{\mathbf{v}} + \nabla\tilde{p} = \mu\Delta\tilde{v} - \tilde{\sigma}\mathbf{g} \\
div(\mathbf{v}) = 0, \tilde{\mathbf{v}}|_{\partial\Omega} = 0
\end{aligned} \tag{2.20}$$

To find a solution to the system with energy estimates and bounds, two variational methods are designed:

i) Eliminate σ by the first equation in (2.20):

$$\Lambda^2\overline{\rho}\tilde{\mathbf{v}} + \Lambda\nabla\tilde{p} = \Lambda\mu\Delta\tilde{v} + \mathbf{g}\partial_{x_3}\overline{\rho}\tilde{v}_3; \;\; div(\tilde{v}) = 0, \;\; \tilde{v}|_{\partial\Omega} = 0, \tag{2.21}$$

To find Λ, design a modified form:

$$\Lambda^2\overline{\rho}\tilde{\mathbf{v}} + \Lambda\nabla\tilde{p} = s\mu\Delta\tilde{v} + \mathbf{g}\partial_{x_3}\overline{\rho}\tilde{v}_3; \;\; div(\tilde{v}) = 0, \;\; \tilde{v}|_{\partial\Omega} = 0, \tag{2.22}$$

There we have the first couple of the energy function and the admissible set by multiplying (2.22) and integrate:

$$\begin{aligned}
E(\tilde{v}) = g\int \partial_{x_3}\overline{\rho}\tilde{v}_3{}^2 dx - s\mu\int |\nabla\tilde{v}|^2 dx \\
\mathbb{A} = \{\tilde{v} \in H_0^1 | \int \overline{\rho}\tilde{\mathbf{v}}^2 dx = 1, div(\tilde{\mathbf{v}}) = 0\}.
\end{aligned} \tag{2.23}$$

And Λ_1 is defined as:

$$\Lambda_1^2 = \sup_{\tilde{\mathbf{v}}\in\mathbb{A}} E(\tilde{\mathbf{v}}). \tag{2.24}$$

Since the energy depends on $s > 0$, we can write more explicitly the square of Λ_1 to be:

$$\alpha(s) := \sup_{\tilde{\mathbf{v}}\in\mathbb{A}} E(\tilde{\mathbf{v}}, s). \tag{2.25}$$

By some investigations of $\alpha(s)$, and the application of the fixed point theorem(as mentioned in the introduction), we can prove that there exists a unique Λ_1 such that $\Lambda_1^2 = \alpha(\Lambda_1)$; moreover, the supremum can be achieved, which means there exists some $\tilde{\mathbf{v}}$, and

$$\Lambda_1^2 = E(\tilde{\mathbf{v}}, \Lambda_1) \tag{2.26}$$

is satisfied; from there we can find a solution $(\tilde{v_0}, \tilde{\rho_0}, \Lambda_1)$ for the problem (2.21).

ii) In the first variational method we proved the existence of a solution to the linearized problem; and in order to proceed further, a few energy estimates are needed. Thus we can construct the second form of the exponential growth rate by first changing the linearized problem into

$$\begin{aligned}
&\Lambda\frac{\tilde{\sigma}}{\partial_{x_3}\overline{\rho}} = -\tilde{v_3}\\
&\frac{(\Lambda\overline{\rho}\tilde{\mathbf{v}} + \nabla\tilde{p} - \mu\Delta\tilde{\mathbf{v}})}{g} = -\tilde{\sigma}e_3\\
&div\tilde{\mathbf{v}} = 0, \tilde{\mathbf{v}}|_{\partial\Omega} = 0.
\end{aligned} \tag{2.27}$$

Multiplying the first equation in (2.27) by $\tilde{\rho}$, second equation by $\tilde{\mathbf{v}}$ and integrate; by summing those two equations we get

$$\Lambda\int(\frac{\tilde{\sigma}^{2\,\prime}}{\overline{\rho}} + \frac{\overline{\rho}|\tilde{\mathbf{v}}|^2}{g})dx = -\int(\frac{\mu|\nabla\tilde{\mathbf{v}}|^2}{g} + 2\tilde{\sigma}\tilde{v_3})dx. \tag{2.28}$$

Then we have the second definition for the pair of the energy functional, and the its admissible set:

$$\begin{aligned}
&\tilde{E}(\tilde{\sigma}, \tilde{\mathbf{v}}) = -\int(\frac{\mu|\nabla\tilde{\mathbf{v}}|^2}{g} + 2\tilde{\sigma}\tilde{v_3})dx,\\
&\tilde{\mathbb{A}} := \{(\tilde{\sigma}, \tilde{\mathbf{v}}) \in L^2 \times H_0^1| \int(\frac{\tilde{\sigma}^{2\,\prime}}{\overline{\rho}} + \frac{\overline{\rho}|\tilde{\mathbf{v}}|^2}{g})dx = 1, div\tilde{\mathbf{v}} = 0\}.
\end{aligned} \tag{2.29}$$

and similar to above, we denote

$$\Lambda_2 := \sup_{(\tilde{\sigma}, \tilde{\mathbf{v}}) \in \tilde{\mathbb{A}}} \tilde{E}(\tilde{\sigma}, \tilde{\mathbf{v}}). \tag{2.30}$$

We can then show that the supremum can be achieved to be a maximum on $\tilde{\mathbb{A}}$; and $\Lambda_1 = \Lambda_2$. Assume $\tilde{\sigma_0}, \tilde{\mathbf{v}}_0$ is a set of the maximizer, then there exists pressure $\tilde{p}_0$ such that $(\tilde{\rho}_0, \tilde{v}_0, \tilde{p}_0)$ solves the above initial boundary problem (2.27).

By employing the two variational forms, we get the sharp exponential growth rate for the solutions $(\sigma, \mathbf{v})$ for the linearized equations controlled by Λ for the following two norms, given initial and boundary conditions described in (2.14):

$$\begin{aligned} \|\sigma(t)\|_{L^2}^2 + \|\mathbf{v}(t)\|_{H^2}^2 &\le Ce^{2\Lambda t}(\|\sigma(0)\|_{L^2}^2 + \|\mathbf{v}(0)\|_{H^2}^2) \\ \|\sigma(t)\|_{L^2}^2 + \|\mathbf{v}(t)\|_{L^2}^2 &\le Ce^{2\Lambda t}(\|\sigma(0)\|_{L^2}^2 + \|\mathbf{v}(0)\|_{L^2}^2). \end{aligned} \tag{2.31}$$

Step 2. Nonlinear estimates

In step one, we have derived for the initial data $(\overline{\sigma}_0, \overline{\mathbf{v}}_0) \in H^2 \times H^2$ given by (2.12), and $div\overline{\mathbf{v}}_0 = 0$, there exists a solution denoted here as $(\sigma^1, \mathbf{v}^1) = e^{\Lambda t}(\overline{\sigma}_0, \overline{\mathbf{v}}_0)$ for the linearlized equations (2.16). For any scaling factor $\delta > 0$, $(\sigma_0^\delta, \mathbf{v}_0^\delta) = \delta(\overline{\sigma}_0, \overline{\mathbf{v}}_0)$, we can deduce that $((\sigma_l^\delta, \mathbf{v}_l^\delta) = \delta(\sigma^1, \mathbf{v}^1)$ is a solution for (2.16) for the linearized solution with the scaled initial data. By the local existence theorem of strong solutions to the Navier-Stokes equation, for the initial data $(\sigma_0^\delta, \mathbf{v}_0^\delta)$, there exits a solution $(\sigma^\delta, \mathbf{v}^\delta)$ to the nonlinearized, perturbed equation (2.12). Denote the difference of the nonlinearized and linearized solution as:

$$(\sigma^d, \mathbf{v}^d) = (\sigma^\delta, \mathbf{v}^\delta) - (\sigma_l^\delta, \mathbf{v}_l^\delta), \tag{2.32}$$

by evaluating (2.12) and (2.16), the difference satisfies:

$$\begin{aligned} \sigma_t^d + \partial_{x_3}\overline{\rho} v_3^d &= -\mathbf{v}^\delta \nabla \sigma^\delta \\ \overline{\rho}\mathbf{v}_t^d - \mu\Delta\mathbf{v}^d + \nabla p^d + \mathbf{g}\sigma^d &= -(\sigma^\delta + \overline{\rho})\mathbf{v}^\delta \cdot \nabla\mathbf{v}^d - \sigma^\delta \mathbf{v}_t^\delta \\ div\mathbf{v}^\delta &= 0, \end{aligned} \tag{2.33}$$

with the initial data and compatibility condition:

$$(\sigma^d(0), \mathbf{v}^d(0)) = (0, \mathbf{0}),\ div\mathbf{v}_0^d = 0. \tag{2.34}$$

By appropriate energy estimates, we derive:

$$\left\|(\sigma^d(t), \mathbf{v}^d(t))\right\|_{L^2}^2 \le C_1\delta^3 e^{3\Lambda t}; \tag{2.35}$$

We define the instability occurrence time to be:

$$T^\delta = \frac{1}{\Lambda} lg \frac{2\epsilon_0}{\delta}, \tag{2.36}$$

where

$$\begin{aligned} m_0 &= min\{\|\overline{\sigma_0}\|_{L^2}, \|\overline{\mathbf{v}_{03}}\|_{L^2}, \|(\overline{\mathbf{v}}_{01}, \overline{\mathbf{v}}_{02})\|_{L^2}\} > 0 \\ \epsilon &= \epsilon_0 m_0, \end{aligned} \tag{2.37}$$

and ϵ_0 is a constant that satisfies

$$\epsilon_0 \le \frac{m_0^2}{8C_1}, \tag{2.38}$$

and C_1 is the constant bound in (2.35). We can show that T^δ does not occur before the breakdown of the classical solution; furthermore, for density σ^δ, we have by (2.35) and

(2.38)

$$\begin{aligned}\left\|\sigma^{\delta}(T^{\delta})\right\|_{L^2} &\geq \left\|\sigma_l^{\delta}(T^{\delta})\right\|_{L^2} - \left\|\sigma^{d}(T^{\delta})\right\|_{L^2} \\ &\geq \delta e^{\Lambda T^{\delta}} \left\|\overline{\sigma}_0\right\|_{L^2} - \sqrt{C_1}\delta^{\frac{3}{2}} e^{3\Lambda\frac{T^{\delta}}{2}} \\ &\geq 2\epsilon_0 \left\|\overline{\sigma}_0\right\|_{L^2} - \sqrt{8C_1}\epsilon_0^{\frac{3}{2}} \\ &\geq \epsilon_0 m_0 = \epsilon\end{aligned} \tag{2.39}$$

Analogously, the nonlinear instability for the velocity can be proved.

2.1.2 Existence for the weak solution of the energy equation

Here we consider the energy equation in (2.1), and we will continue with the existence and the uniqueness of the solutions for the density and the velocity that have already been established in the previous subsection. The following procedures follow from the work by O. A. Ladyzhenskaya, V. A. Solonnikov and N. N. Uraltseva[26]. We firstly introduce some new notations, as we look for a solution of the temperature θ in

$$Q_T = \Omega \times [0, T], \tag{2.40}$$

where T does not exceed the instability occurrence time noted in (2.36). We further denote S_T as the lateral boundary of the cylinder Q_T:$S_T = \partial\Omega \times [0, T]$, and a slice of the cylinder from time t_1 to t_2 as $Q_{t_1,t_2} = \Omega \times [t_1, t_2]$. We use $L_{q,r}(Q_T)$ to denote the Banach space of all functions that have finite norms defined as

$$\|\theta\|_{q,r,Q_T} = (\int_0^T (\|\theta\|_{L^q(\Omega)})^r dt)^{\frac{1}{r}}; \tag{2.41}$$

$W_2^{1,0}(Q_T)$ is the Hilbert space of functions with finite norms defined as

$$\|\theta\|_{W_2^{1,0}(Q_t)} = (\int_{Q_T} |\theta|^2 + |\nabla\theta|^2 dxdt)^{\frac{1}{2}}, \tag{2.42}$$

and $W_2^{1,1}(Q_T)$ is defined similarly as above:

$$\|\theta\|_{W_2^{1,1}(Q_t)} = (\int_{Q_T} |\theta|^2 + |\nabla\theta|^2 + |\theta_t|^2 dxdt)^{\frac{1}{2}}, \tag{2.43}$$

To establish the existence of the solution, we further define the space $V_2(Q_T)$ to be the space of functions in $W_2^{1,0}(Q_T)$ with norm:

$$\|\theta\|_{V_2(Q_T)} := \underset{0\le t\le T}{ess\ sup}\, \|\theta(x,t)\|_{L^2(\Omega)} + \|\nabla\theta\|_{L^2(Q_T)} \tag{2.44}$$

If θ is continuous for t in norm $L^2(\Omega)$, then we say θ belongs to the space $V_2^{1,0}(Q_T)$, and the essential supremum in (2.44) can be replaced by max in the definition for the space of $V_2^{1,0}(Q_T)$; for any $\theta \in V_2^{1,0}(Q_T)$ satisfying

$$\lim_{h\to\infty} \int_0^{T-h} \int_\Omega h^{-1}[\theta(x,t+h) - \theta(x,t)]^2 dxdt = 0, \tag{2.45}$$

we say that θ belongs to the space of $V_2^{1,\frac{1}{2}}(Q_T)$. By a zero on top, for instance, $\overset{0}{V}{}_2^{1,0}(Q_T)$, we mean the functions in the corresponding spaces that vanish on the boundary.

2.1.3 The existence of the weak solution

By dividing both sides of the energy equation by ρ (recall that the density, no matter if it is scaled or not, will remain positive), and by performing integration by part for the diffusion term, we get

$$\theta_t - div(\frac{\kappa}{\rho}\nabla\theta) + (\nabla(\frac{\kappa}{\rho}) + \mathbf{u})\nabla\theta = \frac{\mu}{2\rho}|\nabla\mathbf{u} + \nabla\mathbf{u}^T|^2. \tag{2.46}$$

We will denote from here on:

$$\begin{aligned} \mathbf{b} &= \nabla(\frac{\kappa}{\rho}) + \mathbf{u} \\ f &= \frac{\mu}{2\rho}|\nabla\mathbf{u} + \nabla\mathbf{u}^T|^2. \end{aligned} \tag{2.47}$$

By (2.15), and the fact that we have the positive bounds for the density ρ, there exist positive constants ν, η, such that for any ξ, the uniform parabolicity condition

$$\nu|\xi|^2 \leq \sum_{i,j} a_{ij}\xi_i\xi_j \leq \eta|\xi|^2 \tag{2.48}$$

is satisfied for some positive constants ν, η; here since $a_{ij} = \delta^{ij}\frac{\kappa}{\rho}$, in which δ^{ij} is the Kronecker symbol, this condition is automatically satisfied. We further impose some restrictions on the coefficients:

$$\left\||\mathbf{b}|^2\right\|_{q,r,Q_T} \leq \mu_1, \tag{2.49}$$

for some constant μ_1, with r, q satisfying $\frac{1}{r} + \frac{3}{2q} = 1$, and $q \in (\frac{3}{2}, \infty), r \in [1, \infty)$.
Also for f, there should exist some constant μ_2, such that

$$\|f\|_{q_1,r_1,Q_T} \leq \mu_2, \tag{2.50}$$

where the constants q_1, r_1 meet the condition $\frac{1}{r_1} + \frac{3}{2q_1} = 1 + \frac{3}{4}$, and $q_1 \in [\frac{6}{5}, 2]$, $r_1 \in [1, 2]$.
We will prove in the next section that for suitable ρ and θ, (2.49), (2.50) are satisfied.
We are interested in solving:

$$\begin{aligned} &\theta_t - div(\frac{\kappa}{\rho}\nabla\theta) + \mathbf{b}\cdot\nabla\theta = f \\ &\frac{\partial\theta}{\partial\mathbf{n}} = 0,\ \theta|_{t=0} = \theta_0(x) > 0; \end{aligned} \tag{2.51}$$

We informally derive the suitable form under discussion for the existence of the solution as follows (assuming all terms and functions are smooth enough); multiply (2.51) by a test function $\eta \in W_2^{1,1}(Q_T)$, and integrate from time 0 to T:

$$\int_{Q_T} (\theta_t - div(\frac{\kappa}{\rho}\nabla\theta) + \mathbf{b}\cdot\nabla\theta - f)\eta dxdt = 0 \tag{2.52}$$

Assume for time $t = T$, the test function vanishes, i.e., $\eta(x,T) = 0$; integrate by parts for the first term yields:

$$\begin{aligned}\int_\Omega\int_0^T \theta_t\eta dtdx &= \int_\Omega \frac{d}{dt}(\theta\eta)|_0^T dx - \int_\Omega\int_0^T \theta\eta_t dtdx \\ &= -\int_\Omega \theta(x,0)\eta(x,0)dx - \int_{Q_T}\theta\eta_t dtdx.\end{aligned} \tag{2.53}$$

Also since we impose the Neumann boundary condition on the boundary on S_T,

$$-\int_{Q_T} div(\frac{\kappa}{\rho}\nabla\theta)\eta dxdt = \int_{Q_T}\frac{\kappa}{\rho}\nabla\theta\cdot\nabla\eta dxdt. \tag{2.54}$$

Thus (2.52) can be re-written as:

$$-\int_{Q_T}\theta\eta_t dxdt + \int_{Q_T}(\frac{\kappa}{\rho}\nabla\theta\cdot\nabla\eta + \mathbf{b}\cdot\nabla\theta\eta) - f\eta dxdt = \int_\Omega \theta_0(x)\eta(x,0)dx. \tag{2.55}$$

Denote

$$\begin{aligned}\mathbb{L}_1(\theta,\eta) &= \int_\Omega \frac{\kappa}{\rho}\nabla\theta\cdot\nabla\eta + (\mathbf{b}\cdot\nabla\theta)\eta dx \\ \mathbb{L}_2(f,\eta) &= \int_\Omega(-f\eta)dx;\end{aligned} \tag{2.56}$$

there we see that a natural definition of the weak solution can be presented as follows:

Definition of the weak solution We say that $\theta \in V_2(Q_T)$ is a weak solution to (2.51), if for any $\eta \in W_2^{1,1}(Q_T)$, with $\eta(x,T)=0$, the equation

$$-\int_{Q_T}\theta\eta_t dxdt+\int_0^T(\mathbb{L}_1(\theta,\eta)+\mathbb{L}_2(f,\eta))dt=\int_\Omega\theta_0(x)\eta(x,0)dx \tag{2.57}$$

holds for $\theta_0 \in L^2(\Omega)$.

2.1.4 Energy estimates

We will use the Galerkin Approximation to proceed; in order to do so, we will need estimates of the equations bounded from above; and as a middle step, we first establish a conditional energy estimate in the form of a lemma:

Lemma 2.1.2 *If $\theta \in V_2(Q_T)$, and $\frac{1}{2}\int_\Omega \theta^2(t,x)dx|_{t=t_1}^{t=t_2}+\int_{t_1}^{t_2}[\mathbb{L}_1(\theta,\theta)+\mathbb{L}_2(f,\theta)]dt \le 0$, then $|\theta|_{V_2(Q_T)} \le C[\|\theta(x,0)\|_{L^2(\Omega)}+\|f\|_{q_1,r_1,Q_T}]$, where q_1,r_1 are the constants defined in (2.50).*

Proof. By the uniform parabolic condition and Cauchy's inequality appiled for the $\mathbb{L}_1,\mathbb{L}_2$ terms:

$$\frac{1}{2}\int_\Omega\theta^2dx|_{t=t_1}^{t=t_2}+\nu\int_{t_1}^{t_2}\int_\Omega|\nabla\theta|^2dxdt\le\int_{Q_{t_1,t_2}}\frac{\nu}{2}|\nabla\theta|^2+C|\mathbf{b}|^2|\theta|^2+|f\theta|dxdt, \tag{2.58}$$

thus by merging terms we have

$$\int_\Omega\theta^2dx|_{t=t_1}^{t=t_2}+\nu\int_{t_1}^{t_2}\int_\Omega|\nabla\theta|^2dxdt\le C(\int_{Q_{t_1,t_2}}|\nabla\theta|^2+C|\mathbf{b}|^2\theta^2+|f\theta|dxdt). \tag{2.59}$$

For the first term on the right side of the inequality, denote $\overline{q},\overline{r}$ by $q=\frac{\overline{q}}{\overline{q}-2}$, $r=\frac{\overline{r}}{\overline{r}-2}$, then by the Hölder's inequality

$$\begin{aligned}\int_{Q_{t_1,t_2}}|\mathbf{b}|^2|\theta|^2 &\le C\left\||\mathbf{b}|^2\right\|_{q,r,Q_{t_1,t_2}}\|\theta\|^2_{\overline{q},\overline{r},Q_{t_1,t_2}}\\ &\le \mu(t_1,t_2)\|\theta\|^2_{V_2(Q_{t_1,t_2})}\end{aligned} \tag{2.60}$$

The last inequality comes from the estimate that since $\frac{1}{r} + \frac{3}{2q} = 1$, $\frac{1}{\bar{r}} + \frac{3}{2\bar{q}} = \frac{3}{4}$,

$$\|\theta\|^2_{\bar{q},\bar{r},Q_{t_1,t_2}} \leq C \|\theta\|^2_{V_2(Q_{t_1,t_2})}, \tag{2.61}$$

where the constant C contains the boundary term S_T. Another application of the Hölder's inequality and (2.61) yields:

$$\begin{aligned}\int_{Q_{t_1,t_2}} |f\theta| dxdt \leq C \|f\|_{q_1,r_1,Q_{t_1,t_2}} \|\theta\|_{\bar{q},\bar{r},Q_{t_1,t_2}} &\leq C \|f\|_{q_1,r_1,Q_{t_1,t_2}} \|\theta\|_{V_2(Q_{t_1,t_2})} \\ &=: h(t_1,t_2) \|\theta\|_{V_2(Q_{t_1,t_2})}\end{aligned} \tag{2.62}$$

By combining terms and making use of the definition of $V_2(Q_{t_1,t_2})$, we get

$$\nu_1 \|\theta\|^2_{Q_{t_1,t_2}} \leq \mu(t_1,t_2) \|\theta\|^2_{Q_{t_1,t_2}} + h(t_1,t_2) \|\theta\|_{Q_{t_1,t_2}} + \|\theta(x,t_1)\|^2_{L^2(\Omega)}, \tag{2.63}$$

in which $\nu_1 = min(1,\nu)$. We then assign $t_1 = 0, t_2 = T$. Notice that if $\mu(t_1,t_2) < \nu_1$, by applying Hölder's inequality for the second term on the right, the term on the right hand side $\|\theta\|_{Q_{t_1,t_2}}$ can be "absorbed" by the term on the left, thus the proof is established; then to bypass this obstacle, we do partition for the interval $[t_0 = 0, t_1, ..., t_n = T]$ (assume the partition points are chose so that the norms are valid) of small lengths such that for each $k = 1,2,...,n$,

$$\mu(t_{k-1},t_k) \leq \frac{\nu_1}{2}. \tag{2.64}$$

Therefore, for one small interval $[t_{k-1},t_k]$,

$$\begin{aligned}\frac{\nu_1}{2} \|\theta\|^2_{V_2(Q_{t_{k-1},t_k})} &\leq \|\theta(x,t_{k-1})\|^2_{L^2(\Omega)} + h(t_{k-1},t_k) \|\theta\|_{Q_{t_{k-1},t_k}} \\ &\leq \|\theta(x,t_{k-1})\|^2_{L^2(\Omega)} + \frac{\nu_1}{4} \|\theta\|^2_{V_2(Q_{t_{k-1},t_k})} + Ch^2(t_{k-1},t_k),\end{aligned} \tag{2.65}$$

thus for $k = 0$,

$$\begin{aligned} \|\theta\|^2_{V_2(Q_{t_0,t_1})} &\leq \frac{4}{\nu_1} \|\theta(x,t_0)\|^2_{L^2(\Omega)} + Ch^2(t_0,t_1) \\ &\leq \frac{4}{\nu_1} \|\theta(x,t_0)\|^2_{L^2(\Omega)} + Ch^2(t_0,t_n); \end{aligned} \tag{2.66}$$

to extend the inequality to a larger region, recall by the definition of $\|\theta\|_{V_2(Q_{t_0,t_1})}$,

$$\|\theta(x,t_1)\|^2_{L^2(\Omega)} \leq \|\theta\|^2_{V_2(Q_{t_0,t_1})}; \tag{2.67}$$

and also the inequality (2.66) also holds true for interval $[t_1, t_2]$:

$$\|\theta\|^2_{V_2(Q_{t_1,t_2})} \leq \frac{4}{\nu_1} \|\theta(x,t_1)\|^2_{L^2(\Omega)} + Ch^2(t_0,t_n); \tag{2.68}$$

From there we can see that the right hand side of inequality (2.68) can also be bounded from above by the right hand side of (2.66), differed only by a constant; thus we can extend the region for which inequality (2.66) holds true to $[t_0, t_2]$,and by induction we can extend the region to get the desired estimate for any region in $[0, T]$.

Now assume $\theta \in \overset{0}{V}{}_2^{1,0}(Q_T)$ to be a weak solution of (2.57), and assign $\eta(x,t)$ to be

$$\hat{\eta}_{\overline{h}}(x,t) = h^{-1} \int_{t-h}^{t} \hat{\eta}(x,\tau)d\tau, \tag{2.69}$$

where,$\hat{\eta} \in \overset{0}{W}{}_2^{1,1}(Q_{-h,T})$, and $\hat{\eta}$ equal to zero for $t \geq T-h, t \leq 0$; by changing the integration order of τ and t,

$$\int_0^T \theta(t)\hat{\eta}_{\overline{h}}(t)dt = \int_0^{T-h} \theta_h(t)\hat{\eta}(t)dt, \tag{2.70}$$

in which

$$\theta_h(x,t) = h^{-1} \int_t^{t+h} \theta(x,\tau)d\tau; \tag{2.71}$$

therefore,

$$-\int_{Q_T} \theta \hat{\eta}_{\overline{h}t} dxdt = \int_{Q_T} \theta_h \hat{\eta}_t dxdt. \tag{2.72}$$

with $\hat{\theta}$ vanishing on the lower and upper bounds, integration by parts yields

$$-\int_{Q_T} \theta_h \hat{\eta}_t dxdt = \int_{Q_T} \theta_{ht} \hat{\eta} dxdt. \tag{2.73}$$

Thus if θ is a weak solution of (2.51), by moving the average to the coefficient terms of $\hat{\eta}$, then we get from (2.70), (2.72) that

$$\int_{Q_{T-h}} \theta_{ht}\hat{\eta} + (\frac{\kappa}{\rho}\nabla\theta)_h(\nabla\hat{\eta}) + (\mathbf{b}\cdot\nabla\theta - f)_h\hat{\eta}dxdt = 0 \tag{2.74}$$

Since $\overset{0}{W}{}_2^{1,1}$ dense in $\overset{0}{V}{}_2^{1,0}$, for any $\eta \in \overset{0}{V}{}_2^{1,0}(Q_{-h,T})$, there exists a sub-sequence $\eta_m \in W_2^{1,1}$ that converges to $\eta(x,t)$ strongly in the norm of $V_2^{1,0}(Q_{-h,T})$. By a standard cutoff function argument, for instance if f_k can be described as:

$$f_k(t) = \begin{cases} 0 & \text{for t} \leq 0 \\ 1 & \text{for } \frac{1}{k} \leq t \leq t_1 - \frac{1}{k} \\ 0 & \text{for } t \geq t_1 \end{cases} \tag{2.75}$$

and f_k is continuous elsewhere, then we can define $\hat{\eta}_{m,k}(x,t) = \eta_m(x,t)f_k(t)$. When $m, k \to \infty$, we can pass the limit to acquire:

$$\int_{Q_{t_1}} \theta_{ht}\eta + (\frac{\kappa}{\rho}\nabla\theta)_h(\nabla\eta) + (\mathbf{b}\cdot\nabla\theta - f)_h\eta dxdt = 0 \tag{2.76}$$

for $t_1 \leq T - h$; and here $\eta \in \overset{0}{V}{}_2^{1,0}(Q_T)$; if also θ vanishes on the boundary, then we can take $\eta = \theta_h$ to get:

$$\int_{Q_{t_1}} \theta_{ht}\theta_h + (\frac{\kappa}{\rho}\nabla\theta)_h(\nabla\theta_h) + (\mathbf{b}\cdot\nabla\theta - f)_h\theta_h dxdt = 0. \tag{2.77}$$

Since $\int_{Q_{t_1}} \theta_{ht}\theta_h dxdt = \frac{1}{2}\int_\Omega \theta_h^2(x,t)dx|_{t=0}^{t=t_1}$, by assigning $h \to 0$ we would get the preconditioned form in Lemma 2.1.2, thus $|\theta|_{Q_{t_1}} \leq C[\|\theta(x,0)\|_{L^2(\Omega)} + \|f\|_{q_1,r_1,Q_{t_1}}]$ for $\theta \in \overset{0}{V}{}_2^{1,0}(Q_T)$ and vanishes on the boundary.

For the case that θ belongs to $V_2^{1,0}(Q_T)$, define a continuous function $\zeta(x,t)$ that vanishes on the boundary, and assign $\eta \in \overset{0}{V}{}_2^{1,0}(Q_T)$ by $\eta = \zeta^2\theta_h$ in (2.77); then similarly as above, we can derive the estimates for $\theta \in V_2^{1,0}(Q_T)$ as:

$$|\theta\zeta|_{V_2(Q_T)} \leq C(\|\theta_0\zeta(x,0)\|_{L^2(\Omega)} + \|f\zeta\|_{q_1,r_1,Q_T} + \|\theta\zeta_x\|_{L^2(Q_T)} + \left\|\theta\sqrt{\zeta|\zeta_t|}\right\|_{L^2(Q_T)}). \tag{2.78}$$

2.1.5 Solvability of the energy equation

Recall that

$$\begin{aligned} \mathbb{L}_1(\theta,\eta) &= \int_\Omega \frac{\kappa}{\rho}\nabla\theta\cdot\nabla\eta + (\mathbf{b}\cdot\nabla\theta)\eta dx \\ \mathbb{L}_2(f,\eta) &= \int_\Omega(-f\eta)dx; \end{aligned} \tag{2.79}$$

We proceed by applying Galerkin approximation, which makes use of the concept of separation of variables; assume $\Phi_k(x)$ is a set of orthonormal basis in $L^2(\Omega)$, and each $\Phi_k(x)$ belongs to $W^{1,2}(\Omega)$. Define

$$\Theta^N(x,t) := \sum_{k=1}^N C_k^N(t)\Phi_k(x), \tag{2.80}$$

where each $C_k^N(t)$ is determined by:

$$\frac{d}{dt}(\Theta^N, \Phi_k) + \mathbb{L}_1(\Theta^N, \Phi_k) + \mathbb{L}_2(f, \Phi_k) = 0 \tag{2.81}$$

with initial condition as $C_k^N(0) = (\theta_0, \Phi_k)$, for $k = 1, ..., N$. Then to solve for each coefficient before the basis, we note that (2.81) can be turned into a system of linear ODE:

$$\frac{dC_k^N(t)}{dt} + \sum_i F_i^k C_i^N(t) + G_k(t) = 0, \tag{2.82}$$

for $k = 1, ..., N$. Then by the classical existence theorem of ODE, the above system has a unique solution.

We need to show that the Θ^N constructed can be bounded from above by a constant; we multiply (2.81) by C_k^N and do sum in k to get:

$$\frac{1}{2}\frac{d}{dt}\left\|\Theta^N\right\|_{L^2(\Omega)}^2 + \mathbb{L}_1(\Theta^N, \Theta^N) + \mathbb{L}_2(f, \Theta^N) = 0. \tag{2.83}$$

Integrate to get

$$\frac{1}{2}\left\|\Theta^N\right\|_{L^2(\Omega)}^2 |_{t=0}^{t=t_1} + \int_0^{t_1} (\mathbb{L}_1(\Theta^N, \Theta^N) + \mathbb{L}_2(f, \Theta^N)) = 0 \tag{2.84}$$

Therefore, by Lemma 2.1.2 in previous chapter:

$$\left\|\Theta^N\right\|_{V_2(Q_{t_1})} \leq C(\left\|\Theta^N(x,0)\right\|_{L^2(\Omega)} + \|f\|_{q_1,r_1,Q_{t_1}})). \tag{2.85}$$

Be the definition of basis, $\left\|\Theta^N(x,0)\right\|_{L^2(\Omega)} \leq \|\Phi_0\|_{L^2(\Omega)}$. Thus we have proved that each Θ^N can be bounded from above by a constant. By further defining

$$l_{N,k}(t) = \int \Theta^N(x,t)\Phi_k(x)dx, \tag{2.86}$$

we know by the above calculations and Hölder's Inequality, $l_{N,k}$ are uniformly bounded. From (2.81), integrate from time t to $t + \Delta t$, we would get

$$|l_{N,k}(t+\Delta t) - l_{N,k}(t)| \leq \int_t^{t+\Delta t} |\mathbb{L}_1(\Theta^N, \Phi_k)| + |\mathbb{L}_2(f, \Phi_k)| dt. \tag{2.87}$$

We estimate terms on the right hand side as follows:

$$\begin{aligned}\int_{Q_{t,t+\Delta t}} (\frac{\kappa}{\rho}\nabla\Theta^N)(\nabla\Phi_k) dxdt &\leq C\int_{Q_{t,t+\Delta t}} |\nabla\Theta^N||\nabla\Phi_k| dxdt \\ &\leq C\left\|\nabla\Theta^N\right\|_{L^2(Q_{t,t+\Delta t})}\left\|\nabla\Phi_k\right\|_{L^2(Q_{t,t+\Delta t})} \\ &\leq C\epsilon(\Delta t)\left\|\nabla\Phi_k\right\|_{L^2(\Omega)};\end{aligned} \tag{2.88}$$

Similarly, we have

$$\begin{aligned}\int_t^{t+\Delta t} \mathbb{L}_2(f, \Phi_k) dt &\leq \|f\|_{L^2(Q_{t,t+\Delta t})}\|\Phi_k\|_{L^2(Q_{t,t+\Delta t})} \\ &\leq \epsilon(\Delta t)\|\Phi_k\|_{L^2(\Omega)}; \\ \int_{Q_{t,t+\Delta t}} (\mathbf{b}\nabla\Theta^N)\Phi_k dxdt &\leq \left\||\mathbf{b}|^2\right\|_{q,r,Q_{t,t+\Delta t}}\|\Phi_k\|_{\overline{q},\overline{r},Q_{t,t+\Delta t}}\left\|\nabla\Theta^N\right\|_{L^2(Q_{t,t+\Delta t})} \\ &\leq C\|\Phi_k\|_{\overline{q},\overline{r},Q_{t,t+\Delta t}} \\ &\leq C\|\Phi_k\|_{V_2(Q_{t,t+\Delta t})} \\ &\leq \epsilon(\Delta t)\|\Phi_k\|_{L^2(\Omega)}.\end{aligned} \tag{2.89}$$

Where $\epsilon(\Delta t)$ tends to zero as Δt tends to zero. Summing up we get:

$$\int_t^{t+\Delta t} |\mathbb{L}_1(\Theta^N, \Phi_k)| + |\mathbb{L}_2(f, \Phi_k)| dt \leq \epsilon(\Delta t)(\|\Phi_k\|_{L^2(\Omega)} + \|\nabla\Phi_k\|_{L^2(\Omega)}), \tag{2.90}$$

which implies the equi-continuity of $l_{N,K}(t)$. Thus we can find a subsequence $l_{N_M,k}$ such that for each k fixed, when N_M tends to ∞, $l_{N_M,k}$ converge uniformly to a function $l_k(t)$;

thus we can define the function θ as:

$$\theta = \sum_k l_k(t)\Phi_k(x). \tag{2.91}$$

Now we need to prove that Θ^N converge to θ weakly in $L^2(\Omega)$. For any $\Phi \in L^2(\Omega)$, we write this function in terms of the basis in $L^2(\Omega)$:

$$\Phi = \sum_{k=1}^{\infty}(\Phi, \Phi_k)\Phi_k \tag{2.92}$$

By plugging in the following expression:

$$\begin{aligned}(\Theta^N - \theta, \Phi) &= \sum_{k=1}^{s}(\Phi, \Phi_k)(\Theta^N - \theta, \Phi_k) + (\Theta^N - \theta, \sum_{k=s+1}^{\infty}(\Phi, \Phi_k)\Phi_k) \\ &= \mathbb{I}_1 + \mathbb{I}_2;\end{aligned} \tag{2.93}$$

For any small ϵ, we can make s large enough such that:

$$\mathbb{I}_2 \leq \frac{1}{2}\epsilon; \tag{2.94}$$

Also if s fixed, also notice Φ_k are fixed basis, we can assign N_M large such that $(\Theta^{N_M} - \theta, \Phi_k)$ is arbitrarily small for any t; thus we also have $\mathbb{I}_1 \leq \frac{1}{2}\epsilon$, which implies some subsequence of Θ^{N_M} converge to θ weakly in $L^2(\Omega)$. Also remember that in (2.85) we have the $V_2(Q_T)$ norm of $\{\Theta^N\}$ are bounded by a constant from above; therefore, without changing the subscripts, we can find a sequence $\{\Theta^N\}$ such that: 1. $\Theta^N \to \theta$ weakly in $L^2(Q_T)$, and 2. $\nabla\Theta^N \to \nabla\theta$ weakly in $L^2(Q_T)$ when $N \to \infty$. Also since $\{\Theta^N\}$ bounded, we have $\|\theta\|_{V_2(Q_T)} \leq C$ for some constant C. Thus to prove the first solvability problem, we need to prove that θ constructed satisfies (2.57). For any $H_M = \sum_{k=1}^{M} d_k(t)\Phi_k(t)$, $M \leq N$ and

d_k smooth, and d_k vanishes when $t = T$, we have

$$\begin{aligned}\frac{d}{dt}(\Theta^N, H_M(x,t)) &= \sum_{k=1}^{M}[(\frac{d}{dt}\Theta^N, \Phi_k(x))d_k(t) + \Theta^N, d_k'(t)\Phi_k(x)] \\ &= (\frac{d}{dt}\Theta^N, H^M) + (\Theta^N, H_t^M).\end{aligned} \tag{2.95}$$

Also, multiply (2.81) by $d_k(t)$, for $t = 1, ..., M$ and sum them up while noticing in (2.81), the derivative for the first term falls only on Θ^N, we get

$$(\frac{d}{dt}\Theta^N, H^M) + \mathbb{L}_1(\Theta^N, H^M) + \mathbb{L}_2(f, H^M) = 0 \tag{2.96}$$

Integrate from time 0 to T, notice $d_k(T) = 0$; by applying (2.95), we derive

$$\int_0^T [-(\Theta^N, H_t^M) + \mathbb{L}_1(\Theta^N, H^M) + \mathbb{L}_2(f, H^M)]dt = (\Theta^N(x,0), H^M(x,0)) \tag{2.97}$$

where the round bracket denotes the $L^2(\Omega)$ inner product. Thus for M fixed, since $\{\Theta^N\}$ converges weakly to θ, and $\{\nabla\Theta^N\}$ to $\nabla\theta$, let $N \to \infty$ to get:

$$\int_0^T [-(\theta, H_t^M) + \mathbb{L}_1(\theta, H^M) + \mathbb{L}_2(f, H^M)]dt = (\theta(x,0), H^M(x,0)) \tag{2.98}$$

Since such $H^M = \sum_{k=1}^{M} d_k(t)\Phi_k(t)$ is dense in the space of $W_2^{1,1}(Q_T)$, we have proved that under the definition given in (2.57), a weak solution exists.

2.1.6 Uniqueness of the solution

To prove the uniqueness of the weak solution, we will further demonstrate θ constructed as above actually is not only in $V_2(Q_T)$, but it is also in the space of $V_2^{1,0}(Q_T)$ with norm:

$$\|\theta\|_{V_2^{1,0}(Q_T)} := \underset{0\le t\le T}{max} \|\theta(x,t)\|_{L^2(\Omega)} + \|\nabla\theta(x,t)\|_{L^2(\Omega)}. \tag{2.99}$$

In the first place, we extend the range of the functions in (2.57):

$$\theta^\star(x,t) := \begin{cases} \theta(x,t) & \text{for t} \in [0,T] \\ \theta(x,-t) & \text{for t} \in [-T,0) \\ 0 & \text{elsewhere} \end{cases} \tag{2.100}$$

And denote:

$$\begin{aligned} -\frac{\kappa}{\rho}\nabla\theta &= \mathbf{T}_1(x,t) \\ -\mathbf{b}\cdot\nabla\theta &= \mathbb{T}_2(x,t) \end{aligned} \tag{2.101}$$

Then (2.55) can be rewritten as:

$$\int_{Q_T} -\theta\eta_t dxdt - \int_\Omega \theta_0(x)\eta(x,0)dx = \int_{Q_T} \mathbf{T}_1\nabla\eta + (\mathbb{T}_2 + f)\eta dxdt \tag{2.102}$$

Then by definition (2.101), we can extend those two functions above, along with f, also to the region $\infty, -\infty$ in t:

$$\mathbf{T}_1^\star(x,t) := \begin{cases} \mathbf{T}_1(x,t) & \text{for t} \in [0,T] \\ -\mathbf{T}_1(x,t) & \text{for t} \in [-T,0) \\ 0 & \text{elsewhere,} \end{cases} \tag{2.103}$$

$$\mathbb{T}_2^\star(x,t) := \begin{cases} \mathbb{T}_2(x,t) & \text{for t} \in [0,T] \\ -\mathbb{T}_2(x,-t) & \text{for t} \in [-T,0) \\ 0 & \text{elsewhere,} \end{cases} \tag{2.104}$$

$$f^\star(x,t) := \begin{cases} f(x,t) & \text{for t} \in [0,T] \\ -f(x,-t) & \text{for t} \in [-T,0) \\ 0 & \text{elsewhere.} \end{cases} \tag{2.105}$$

And by examining each term, we get:

$$\int_{Q_{-\infty,\infty}} -\theta^\star \eta_t dxdt = \int_{-\infty}^{0} -\theta^\star \eta_t dxdt + \int_{0}^{\infty} -\theta^\star \eta_t dxdt \tag{2.106}$$

in which

$$\begin{aligned} \int_{-\infty}^{0} -\theta^\star \eta_t dxdt &= \int_{\infty}^{0} -\theta(x,\tau)(-\eta_\tau)dx(-d\tau) \\ &= \int_{0}^{\infty} \theta(x,\tau)\eta_\tau dxd\tau \end{aligned} \tag{2.107}$$

Also we get

$$\begin{aligned} \int_{-\infty}^{0} \mathbf{T}_1^\star(x,t)\nabla\eta dxdt &= \int_{-\infty}^{0} -\mathbf{T}_1(x,-t)\nabla\eta dxdt \\ &= -\int_{0}^{\infty} \mathbf{T}_1(x,\tau)dxd\tau; \end{aligned} \tag{2.108}$$

Similarly, $\int_{-\infty}^{0} f^\star(x,t)\eta dxdt = -\int_0^\infty f\eta dxdt$. Therefore by combining the above results, we would get a new expression without the initial condition term from (2.102):

$$\int_{Q_{-\infty,\infty}} -\theta^\star \eta_t dxdt = \int_{Q_{-\infty,\infty}} \mathbf{T}_1^\star \nabla\eta + (\mathbb{T}_2^\star + f^\star)\eta dxdt. \tag{2.109}$$

In the form above η is any test function from $W_2^{1,1}(Q_{-\infty,\infty})$, with $\eta(t) = 0$ for $t \geq T$. Take a smooth cutoff function w defined by:

$$w = \begin{cases} 1 & \text{t} \in [-T+\delta, T-\delta], \\ 0 & |t| \geq T \end{cases} \tag{2.110}$$

and $\eta(x,t) = w(t)\Phi(x,t)$, where $\Phi(x,t) \in W_2^{1,1}(Q_{-\infty,\infty})$, and denote:

$$v = \theta^\star(x,t)w(t). \tag{2.111}$$

Then the above (2.109) can be rephrased after combining terms:

$$\begin{aligned}\int_{Q_{-\infty,\infty}} -v\Phi_t dxdt &= \int_{Q_{-\infty,\infty}} \mathbf{T}_1^\star w \cdot \nabla\Phi + (\mathbb{T}_2^\star w + f^\star w + \theta^\star w_t)\Phi dxdt \\ &:= \int_{Q_{-\infty,\infty}} (\mathbf{A}\nabla\Phi + B\Phi) dxdt\end{aligned} \tag{2.112}$$

where from the properties of the weak solution and the coefficients, we get $\mathbf{A} \in L^2(Q_{-\infty,\infty})$, $B \in L_{q_1,r_1,Q_{-\infty,\infty}}$, in which q_1, r_1 satisfies (2.51). Then since $\Phi \in W_2^{1,1}(Q_{-\infty,\infty})$, we can take

$$\Phi = h^{-1}\int_{t-h}^{t} \eta(x,\tau)d\tau, \tag{2.113}$$

in which $\eta \in W_2^{1,1}(Q_{-\infty,\infty})$. By the averaging technique introduced in section 2.1.4, and by (2.70), (2.72), (2.73), the average can be shifted from η onto other parts in (2.112):

$$\int_{Q_{-\infty,\infty}} -v_h\eta_t dxdt = \int_{Q_{-\infty,\infty}} \mathbf{A}_h\nabla\eta + B_h\eta dxdt, \tag{2.114}$$

where v_h is defined in (2.69). Here motivated by separation of variable, we assign

$$\eta(x,t) = r(t)\varphi(x); \tag{2.115}$$

with $r(t)$ smooth and $\varphi(x) \in W_1^{1,1}(\Omega)$. Then by plugging into (2.114):

$$\begin{aligned}\int_{-\infty}^{\infty} -r'(\int_\Omega v_h\varphi dx)dt &= \int_{Q_{-\infty,\infty}} \mathbf{A}_h r \cdot \nabla\varphi + B_h r\varphi dxdt \\ &= \int_{-\infty}^{\infty} r(\int_\Omega \mathbf{A}_h\nabla\varphi + B_h\varphi dx)dt.\end{aligned} \tag{2.116}$$

Since r can be any smooth function, by the definition of weak derivative, we have

$$\frac{d}{dt}(v_h, \varphi)_{L^2(\Omega)} = (\mathbf{A}_h, \nabla\varphi)_{L^2(\Omega)} + (B_h, \varphi)_{L^2(\Omega)}. \tag{2.117}$$

Since φ is only a function with respect to x, and recall by definition, the weak derivative of v_h exists, then we can also write (2.117) as

$$(v_{ht}, \varphi)_{L^2(\Omega)} = (\mathbf{A}_h, \nabla\varphi)_{L^2(\Omega)} + (B_h, \varphi)_{L^2(\Omega)}. \tag{2.118}$$

Thus for any $h_1, h_2 > 0$, choose $\varphi = v_{h_1} - v_{h_2}$, we get:

$$\begin{aligned}((v_{h_1} - v_{h_2})_t, v_{h_1} - v_{h_2}) &= \frac{1}{2}\frac{d}{dt}\left\|v_{h_1} - v_{h_2}\right\|^2_{L^2(\Omega)} \\ &= (\mathbf{A}_{h_1} - \mathbf{A}_{h_2}, \nabla(v_{h_1} - v_{h_2})) + (B_{h_1} - B_{h_2}, v_{h_1} - v_{h_2});\end{aligned} \tag{2.119}$$

integrate the above equality from any time t_1 to t_2 to get:

$$\frac{1}{2}\left\|v_{h_1} - v_{h_2}\right\|^2_{L^2(\Omega)}\Big|^{t=t_2}_{t=t_1} = \int_{t_1}^{t_2} (\mathbf{A}_{h_1} - \mathbf{A}_{h_2}, \nabla(v_{h_1} - v_{h_2})) + (B_{h_1} - B_{h_2}, v_{h_1} - v_{h_2})dt. \tag{2.120}$$

By Hölder's inequality, the right hand side is bounded by

$$\begin{aligned}&\int_{t_1}^{t_2} (\mathbf{A}_{h_1} - \mathbf{A}_{h_2}, \nabla(v_{h_1} - v_{h_2})) + (B_{h_1} - B_{h_2}, v_{h_1} - v_{h_2})dt \leq \\ &C(\|\mathbf{A}_{h_1} - \mathbf{A}_{h_2}\|^2_{L^2(Q_{t_1,t_2})} + \|\nabla(v_{h_1} - v_{h_2})\|^2_{L^2(Q_{t_1,t_2})} + \\ &\|v_{h_1} - v_{h_2}\|^2_{q_1',r_1',Q_{t_1,t_2}} + \|B_{h_1} - B_{h_2}\|^2_{q_1,r_1,Q_{t_1,t_2}});\end{aligned} \tag{2.121}$$

from the lemma proven by O. A. Ladyzhenskaya, V. A. Solonnikov and N. N. Uraltseva, we have:

Corollary 2.1.2.1 *If $\theta \in V_2^{1,0}(Q_T)$, then $\theta_h(x,t)$ defined in (2.69) is in $W_2^{1,1}(Q_{T-\delta})$, and*

$$\lim_{h\to 0} |\theta_h - \theta|_{Q_{T-\delta}} = 0. \tag{2.122}$$

Thus when $h_1, h_2 \to 0$, the right hand side of (2.121) converges to zero. So if we fix $t_1 = -\infty$, for any t_2, (2.120) still holds, thus we get when $h \to 0$, $v_h \to v$ in $L^2(\Omega)$; also by the above expression we can observe that the converge is uniform in t; thus v is continuous with respect to the L^2 norm of v. Then since $v = \theta(x,t)w(t)$, with w(t) defined in (2.110), we would also get θ is continuous with respect to the L^2 norm. Thus be the fact that θ already belonging to $V_2(Q_{T-\delta})$, we get $\theta \in V_2^{1,0}(Q_{T-\delta})$.

Therefore, by the above discussion, we can formulate our uniqueness theorem for the weak solution of the problem:

Theorem 2.1.3 *Assume θ is a weak solution to the problem (2.51) defined by (2.57), with conditions (2.48), (2.49), (2.50) satisfied; then θ is unique in the space of $V_2^{1,0}(Q_T)$.*

Proof. Assume there exists two solutions, $\theta_1(x,t), \theta_2(x,t)$ to the problem (2.51) with the same initial condition satisfying (2.57); then

$$-\int_{Q_T} (\theta_1 - \theta_2)\eta_t dxdt + \int_0^T (\mathbb{L}_1(\theta_1 - \theta_2, \eta) + \mathbb{L}_2(f, \eta))dt = f_1 - f_2 = 0; \tag{2.123}$$

with the initial condition $\theta_1(x,0) - \theta_2(x,0) = 0, x \in \Omega$. Since we have proved that any weak solution belong to $V_2(Q_T)$ is also a member of $V_2^{1,0}(Q_T)$, then we can take $\eta = \theta_1 - \theta_2 \in V_2^{1,0}(Q_T)$ and derive by (2.78) that

$$|(\theta_1 - \theta_2)\zeta|_{V_2(Q_T)} \le C(\|(\theta_1 - \theta_2)\zeta_x\|_{L^2(Q_T)} + \left\|(\theta_1 - \theta_2)\sqrt{\zeta|\zeta_t|}\right\|_{L^2(Q_T)}). \tag{2.124}$$

For any continuous function $\zeta \in \overset{0}{W}{}_2^{1,1}(Q_T)$. Notice that the non-homogeneous part vanishes on the right hand side. Thus $\theta_1 = \theta_2$, and we have the uniqueness theorem.

2.2 Regularity of the solution based on the conditions of density and velocity

After the construction of the weak solution, we turn to the regularity of the solution, given suitable conditions for $\rho, \mathbf{u}$. We will improve the regularity of θ in the Hölder continuous spaces, thus we need some more definitions on Hölder space for two variables:

2.2.1 Sobolev space and Hölder continuous space for two variables

For a region $Q_T = \Omega \times [0, T]$, integers $q \geq 1$, $l \geq 1$, we say a function u is in the sobolev space $W_q^{2l,l}(Q_T)$, if all the partial derivatives $\partial_t^r \partial_x^s$ exists for $2r + s \leq 2l$, and each norm is in $L^q(Q_T)$; we define its norm to be:

$$\|u\|_{W_q^{2l,l}(Q_T)} = \sum_{i=0}^{2l} \sum_{2r+s=i} \|\partial_t^r \partial_x^s u\|_{L^q(Q_T)} . \tag{2.125}$$

As a special case, when $l = 1$, we find all pairs of (r, s) such that $2r + s \leq 2$, and that means

$$\|u\|_{W_q^{2,1}(Q_T)} = \|u\|_{L^q(Q_T)} + \|Du\|_{L^q(Q_T)} + \left\|D^2 u\right\|_{L^q(Q_T)} + \|u_t\|_{L^q(Q_T)} . \tag{2.126}$$

We define the Hölder continuous space for two variables similarly; we say that u is in the space $H^{l,\frac{l}{2}}(\overline{Q_T})$ for $l > 0$, if u in continuous in $\overline{Q_T}$, and all derivatives exists for $\partial_t^r \partial_x^s$ where $2r + s < l$, and the norm $|u|_{Q_T}^{(l)}$ is defined as:

$$|u|_{Q_T}^{(l)} = < u >_{Q_T}^{(l)} + \sum_{j=0}^{[l]} < u >_{Q_T}^{(j)} . \tag{2.127}$$

From here on we always use "l" to denote the fractional part of the Hölder norm, where "j" is used for the integral part, for $j = 0$:

$$< u >_{Q_T}^{(0)} = |u|_{Q_T}^{(0)} = \max_{Q_T} |u|, \tag{2.128}$$

and for the integer part, for each $j \in [0, [l]]$, where $[l]$ denotes the largest integer less than or equal to l, we have

$$< u >^{(j)}_{Q_T} = \sum_{2r+s=j} |\partial_t^r \partial_x^s u|^{(0)}_{Q_T}. \tag{2.129}$$

For the non-integral part, $< u >^{(l)}_{Q_T} = < u >^{(l)}_{x,Q_T} + < u >^{(\frac{l}{2})}_{t,Q_T}$, in which

$$\begin{aligned} < u >^{(l)}_{x,Q_T} &= \sum_{2r+s=[l]} < \partial_t^r \partial_x^s u >^{(l-[l])}_{x,Q_T} \\ < u >^{(\frac{l}{2})}_{t,Q_T} &= \sum_{\frac{l}{2}-\frac{2r+s}{2} \in (0,1)} < \partial_t^r \partial_x^s u >^{(\frac{l}{2}-\frac{2r+s}{2})}_{t,Q_T} . \end{aligned} \tag{2.130}$$

2.2.2 Smoothness conditions for the coefficients of the energy equation

Since the coefficients of the energy equation depends on $\rho, \mathbf{u}$, we need some regularity so that the existence of the weak solution and further regularity results can be obtained. There are many significant relevant results for the Navier Stokes equation given in various forms [9, 36, 39]; here we mainly cite the work by Ladyhzenskaya and Solonnikov [28]; we consider in three dimensional space a nonhomogeneous incomrpressible fluid system with viscosity:

$$\begin{aligned} \rho_t + div(\rho \mathbf{u}) &= 0 \\ (\rho u)_t + div(\rho \mathbf{u} \otimes \mathbf{u}) + \nabla P &= \mu \Delta \mathbf{u} + \rho \mathbf{f} \\ div \mathbf{u} &= 0; \end{aligned} \tag{2.131}$$

in $\Omega \times (0, T)$, given initial data $(\rho(x, 0), \mathbf{u}(x, 0)) = (\rho_0, \mathbf{u}_0)$ satisfying the conditions:

$$\rho_0 \in C^1(\overline{\Omega}), \inf_{x \in \Omega} \rho_0(x) > 0, u_0 \in W^{2-\frac{2}{r},r}(\Omega), div u_0 = 0 \tag{2.132}$$

for $r > 3$; then there exists a unique solution $(\rho, \mathbf{u}, P)$ to the above equations and a time $T > 0$ such that:

$$\begin{aligned} &\mathbf{u} \in L^r(0,T;W^{2,r}(\Omega)), \mathbf{u}_t \in L^r(Q_T); \\ &\rho \in C^1(\overline{Q_T}). \end{aligned} \tag{2.133}$$

Notice since $\mathbf{u}_t \in L^r(Q_T)$, then by the above discussion on Sobolev spaces involving two variables and (2.126), we have $\mathbf{u} \in W_r^{2,1}(Q_T)$; by the embedding theorem from two variables Sobolev spaces into Hölder continuous spaces:

$$W_r^{2,1}(Q_T) \hookrightarrow \begin{cases} H^{\alpha,\frac{\alpha}{2}}(\overline{Q_T}) \quad 0 < \alpha \le 2 - \frac{n+2}{r}, \text{ for } \frac{n+2}{2} < r < n+2 \\ H^{\alpha,\frac{\alpha}{2}}(\overline{Q_T}) \quad 0 < \alpha < 1, \text{ for } r = n+2 \\ \text{H}^{1+\alpha,\frac{1+\alpha}{2}}(\overline{Q_T}) 0 < \alpha < 1 - \frac{n+2}{r}, \text{ for } r > n+2 \end{cases} \tag{2.134}$$

in which n denotes the dimension of space variables, and

$$H^{1+\alpha,\frac{1+\alpha}{2}}(\overline{Q_T}) = \{u \in C^{(1+\alpha)/2}(\overline{Q_T}), and \frac{\partial u}{\partial x_i} \in H^{\alpha,\frac{\alpha}{2}}(\overline{Q_T})\}. \tag{2.135}$$

therefore, when r = 6, we have from the embedding theorem above,

$$\mathbf{u} \in H^{1+l,\frac{1+l}{2}}(\overline{Q_T}) \tag{2.136}$$

for $l = \frac{1}{6}$.

Conditions for the existence of the weak solution.

We need to verify that under (2.136), (2.49) and (2.50) are fulfilled. In view of (2.49), when $r, q = 2, 2$, $\frac{1}{r} + \frac{3}{2q} = 1$; since $\mathbf{b} = \nabla(\frac{\kappa}{\rho}) + \mathbf{u}$, by (2.133) and (2.136), we have established that $\mathbf{b}$ will be continuous in $\overline{Q_T}$; therefore, given the region Ω bounded, condition (2.49) is satisfied for some constant μ_1. Also for

$$f = \frac{\mu}{2\rho}|\nabla \mathbf{u} + \nabla \mathbf{u}^T|^2, \tag{2.137}$$

we have $\frac{\partial \mathbf{u}}{\partial x_i} \in H^{l,\frac{l}{2}}(\overline{Q}_T)$ by (2.135); thus f is also continuous in $\overline{Q_T}$, thus in (2.50), we can set $r_1 = 1, q_1 = 2$ to fulfill the condition.

We should remark here that we can loosen the condition for $\rho, \mathbf{u}$ for the weak solution; however, since we are about to prove more regularity results for θ, we will continue with the same conditions (2.133) as mentioned above.

2.2.3 New notations, the compatibility condition and zero initial data condition

We now write back the energy equation as the non-divergence form:

$$\begin{aligned} &\theta_t + \mathbf{u} \cdot \nabla\theta - \frac{\kappa}{\rho}\Delta\theta = f \\ &\nabla\theta \cdot \mathbf{n}|_{S_T} = 0 \\ &\theta(x, 0) = \theta_0 > 0. \end{aligned} \tag{2.138}$$

We introduce the notation

$$\theta_{(k)} = \frac{\partial^k \theta}{\partial t^k}|_{t=0}, \tag{2.139}$$

thus $\theta_{(0)} = \theta_0(x), \theta_{(1)} = \frac{\kappa}{\rho}\Delta\theta_0(x) - \mathbf{u}\nabla\theta_0 + f(x, 0)$; we call the compatibility condition is satisfied if for $l < \frac{1}{2}$,

$$\nabla\theta \cdot \mathbf{n}|_{t=0} = \nabla\theta_0 \cdot \mathbf{n} = 0. \tag{2.140}$$

Notations of operators, and the requirement for Ω.

For the sake of convenience, we introduce several notations for the proof. We define $\mathbb{L}, \mathbb{B}$

to be the operators:

$$\begin{aligned} &\mathbb{L}(x,t,\frac{\partial}{\partial x},\frac{\partial}{\partial t}) = \frac{\partial}{\partial t} - \frac{\kappa}{\rho}\Delta + \mathbf{u}\cdot\nabla \\ &\mathbb{B} = \mathbf{n}\cdot\nabla \end{aligned} \tag{2.141}$$

Thus we are interested in solving:

$$\begin{aligned} \mathbb{L}\theta &= f \\ \theta|_{t=0} &= \theta_0 \\ \mathbb{B}\theta|_{S_T} &= 0 \end{aligned} \tag{2.142}$$

in $Q_T = \Omega \times (0,T)$.

Definition of spaces We define spaces:

$$\begin{cases} \mathbb{R}_{n+1}, \text{for } (\mathbf{x},t); \\ \mathbb{R}_{n-1}, \text{for } x' = (x_1,...,x_{n-1}); \\ \mathbb{D}_{n+1}, \text{for } (\mathbf{x},t), when\ t>0; \\ \mathbb{D}_n, \quad \text{for } (\mathbf{x}), when\ x_n>0; \\ \tilde{\mathbb{D}}_{n+1}, \text{ for } (\mathbf{x},t), when\ x_n>0; \\ \tilde{\mathbb{D}}_n, \text{ for } (\mathbf{x},t), when\ x_n=0, t>0; \\ R, \quad \text{for } (\mathbf{x},t), when\ x_n>0, t>0 \end{cases} \tag{2.143}$$

Also we denote a space $S^{(T)}$ that is associated with any space S above, and satisfies $t<T$.

Requirements for the domain Ω At each point of the boundary of Ω, one requirement is that we can establish a local coordinate system with a ball centered at $\xi \in S$, within a radius of d, and the surface is given by $y_n = F(y_1,...,y_{n-1})$.We call the boundary is in the space of H^l, if $F(y') \in H^l(B(\xi, |y'| \leq \frac{d}{2}))$ where $y' = (y_1,...,y_{n-1})$, and for any ξ on the boundary and if they can be bounded by the same constant.

Intuitively to enhance the regularity, we need to divide the region into small parts such that some of them are restricted by the boundary condition, whereas others are interior regions, and by piecing them together in some appropriate method, we can establish the existence of the solution in a certain desired class. More precisely, for any small $\lambda > 0$, we require there exists such a separation of the domain Ω:

1). There are two sets of sub-domains $\{\omega^k\}$ and $\{\Omega^k\}$, both finite, and for each k,

$$\omega^k \subset \Omega^k \ , \ \text{and} \ \underset{k}{\cup} \omega^k = \underset{k}{\cup} \Omega^k = \Omega; \tag{2.144}$$

2). For any $x \in \Omega$, by 1) it must belong to some ω^k; we furthermore require that $d(x, \Omega^k \backslash \omega^k) \geq d\lambda$, where d is the constant defined above;

3). There can at most be N_0 randomly chosen subsets of Ω^k such that their intersection is non-empty;

4). Interior small regions. We denote the indexes of the set $\{\Omega^k\}$, in which each Ω^k is separated from the boundary S by R_{in} (meaning inside the domain). Also, we assume that for such fixed k, any pair Ω^k, ω^k, there exists a center $\xi^k \in \Omega$, such that both Ω^k, ω^k can be described as cubes of linear dimension $2\beta\lambda$ and $\beta\lambda$ respectively;

5). Adjacent small regions. We denote the indexes where Ω^k, ω^k touches the boundary by R_{on}; for each k, we assume we can use the local coordinate system $\{y_i\}_{i=1}^n$ at $\xi^k \in S$ and define the region by

$$\begin{aligned} &|y_i| < \lambda, \ \text{ for } i = 1, ..., n-1; \ 0 < y_n - F(y') < 2\lambda \ \text{ for } \Omega^k; \\ &|y_i| < \frac{\lambda}{2}, \ \text{ for } i = 1, ..., n-1; \ 0 < y_n - F(y') < \lambda \ \text{ for } \omega^k; \end{aligned} \tag{2.145}$$

By a change of variable

$$z_i = y_i, \ \ i = 1, ..., n-1, \ \ z_n = y_n - F(y'), \tag{2.146}$$

the region can be further written as

$$|z_i| < \lambda, 0 < z_n < 2\lambda \tag{2.147}$$

for Ω^k, and we denote this domain by R.

6). Cutoff and weight functions. For each k, we define the smooth function $\zeta^k(x)$ having the properties $0 \le \zeta^k(x) \le 1$, and $|D^s\zeta^k| \le \frac{C_s}{\lambda^s}$; also

$$\zeta^k(x) = \begin{cases} 1 \ \text{ for } x \in \omega^k; \\ 0 \ \text{ for } x \in \Omega \backslash \Omega^k. \end{cases} \tag{2.148}$$

By the above property 3), for any x in Ω,

$$1 \le \sum_k \zeta^k(x) \le N_0, \tag{2.149}$$

and the weight function defined as

$$\eta^k(x) = \frac{\zeta^k(x)}{\sum_i [\zeta^i(x)]^2} \tag{2.150}$$

has the property $|D^s\eta^k| \le \frac{C_s}{\lambda^s}$, and it is also clear that

$$\sum_k \zeta^k \eta^k = 1 \tag{2.151}$$

Zero initial data condition. We will prove a regularity theorem first for a special type of functions: We say that a function $f \in H^{l,\frac{l}{2}}(\overline{Q_T})$ has zero initial condition on Q_T, if

$$\frac{\partial^k f}{\partial t^k}|_{t=0} = 0, \ \text{ for } k = (0, ..., [\frac{l}{2}]); \tag{2.152}$$

and we denote this space by $\underset{0}{H}^{l,\frac{l}{2}}(\overline{Q_T})$. Here for instance when $l = \frac{1}{6}$, then the required initial zero condition would be $f(x,0) = 0$.

Constant coefficient problems in the half space and straightened boundary spaces. Consider the problem in $\mathbb{D}^{(T)}_{n+1}$:

$$\frac{\partial \theta_1}{\partial t} - \sum a_{ij} \frac{\partial^2 \theta_1}{\partial x_i \partial x_j} = f, \tag{2.153}$$

$f \in \underset{0}{H}^{l,\frac{l}{2}}(\overline{D^{(T)}_{n+1}})$, a_{ij} constants. Also the problem in $R^{(T)} = \{(\mathbf{x}, t) \in \mathbb{R}_{n+1}, x_n > 0, t > 0\}$:

$$\begin{aligned} &\frac{\partial \theta_2}{\partial t} - \sum a_{ij} \frac{\partial^2 \theta_2}{\partial x_i \partial x_j} = f, \\ &\mathbf{b} \cdot \nabla \theta_2|_{x_n=0} = \Phi(x', t), \end{aligned} \tag{2.154}$$

where a_{ij}, b_i are also constants, $f \in \underset{0}{H}^{l,\frac{l}{2}}(\overline{R^{(T)}})$, $\Phi \in \underset{0}{H}^{l+1,\frac{l+1}{2}}(\overline{\tilde{D_n}^{(T)}})$. Then if

$$a_{ij} = \delta^{ij}, \tag{2.155}$$

By the standard theorems for the heat equation, it is known that unique solutions for the above two initial/boundary value problems exist with estimates:

$$\begin{aligned} &< \theta_1 >^{(l+2)}_{D^{(T)}_{n+1}} \le C < f >^{(l)}_{D^{(T)}_{n+1}}; \\ &< \theta_1 >^{(l+2)}_{R^{(T)}} \le C(< f >^{(l)}_{R^{(T)}} + < \Phi >^{(l+1)}_{\tilde{D_n}^{(T)}}); \end{aligned} \tag{2.156}$$

When a_{ij} not delta-function, we make use of the transformation of the coordinates, $\mathbf{y} = P\mathbf{x}$ where P is an orthogonal matrix, and $A = a_{ij}$ has eigenvalues $\{\lambda_i\}$ to translate the system into

$$\frac{\partial \theta_1}{\partial t} - \sum \lambda_i \frac{\partial^2 \theta_1}{\partial y_i^2} = f(y,t). \tag{2.157}$$

Another scaling factor $z_i = \frac{y_i}{\sqrt{\lambda_i}}$ would result in the heat equation; then finally we use

$\xi = Q\mathbf{z}$, Q another orthogonal transformation, to translate the surface into $\xi_n = 0$, and then we can solve the problem just as the heat equation above.

Statement of theorems. Lastly for this subsection, we state the regularity theorems:

Theorem 2.2.1 (Regularity theorem in a Hölder continuous class) *For the system of equations*

$$\begin{aligned}
&\mathbb{L}(x,t,\frac{\partial}{\partial x},\frac{\partial}{\partial t})\theta(x,t) = f(x,t) \\
&\theta(x,0) = \theta_0 \qquad\qquad (2.158)\\
&\boldsymbol{n}\cdot\nabla\theta|_{S_T} = \Phi(x,t)
\end{aligned}$$

with coefficients of $\mathbb{L}$ *belonging to* $H^{l,\frac{l}{2}}(\overline{Q_T}), f \in H^{l,\frac{l}{2}}(\overline{Q_T}), \Phi \in H^{l+1,\frac{l+1}{2}}(\overline{S_T})$. *Also assume boundary* $S \in H^{l+2}$, $\theta_0 \in H^{l+2}(\Omega)$ *and the compatibility condition is satisfied; then the problem has a unique solution from the class* $\boldsymbol{H}^{l+2,\frac{l}{2}+1}(\overline{Q_T}), and |\theta|_{Q_T}^{(l+2)} \leq C(|f|_{Q_T}^{(l)} + |\theta_0|_{\Omega}^{(l+2)} + |\Phi|_{S_T}^{(l+1)})$.

Note that when $\Phi = 0$, we have the regularity theorem for the Neumann boundary condition. To prove this theorem, we will need the middle step:

Theorem 2.2.2 (Regularity theorem for zero initial condition) *Consider the system (2.158), and if furthermore,* $f \in \underset{0}{H}^{l,\frac{l}{2}}(\overline{Q_T})$, $\Phi \in \underset{0}{H}^{l+1,\frac{l+1}{2}}(\overline{S_T})$, *then we can find a solution* θ *in the class of* $\underset{0}{H}^{l+2,\frac{l}{2}+1}(\overline{Q_\tau})$ *for some* $\tau < T$, *with estimate* $|\theta|_{Q_\tau}^{(l+2)} \leq C(|f|_{Q_\tau}^{(l)} + |\Phi|_{S_\tau}^{(l+1)})$.

2.2.4 Several results and lemmas

First we present some extension results. Assume in the first place Ω is a bounded subset in R_n), with boundary in the class of H^l; then it is known that we can extend a function $f(x) \in H^l(\Omega)$ to a function $f^\star$ on the whole space and

$$\|f^\star\|_{H^l(\mathbb{R}_n)} \leq C\,\|f\|_{H^l(\Omega)}\,, \qquad (2.159)$$

and $f^\star = f$ on Ω; Also we can extend a function defined in smaller dimensional spaces to higher dimensional spaces:

Theorem 2.2.3 (Extension from smaller space to larger space) *If for $j = 0, ..., [\frac{l}{2}]$, $\phi_j(x) \in H^{(l-2j)}(\overline{\mathbb{R}_n})$, and l is a positive non-integral; then there exists $u \in H^{l,\frac{l}{2}}(D_{n+1}^{(T)})$, such that*

$$\frac{\partial^j u}{\partial t^j}|_{t=0} = \phi_j(x). \tag{2.160}$$

Moreover, $|u|_{D_{n+1}^{(T)}}^{(l)} \leq C \sum_{j=0}^{[\frac{l}{2}]} |\phi_j|_{\mathbb{R}_n}^{(l-2j)}$, and

$$< u >_{D_{n+1}}^{(l)} \leq C \sum_{j=0}^{[\frac{l}{2}]} < \phi_j >_{\mathbb{R}_n}^{(l-2j)} . \tag{2.161}$$

Proof. When $l > 2$, we define $u(x,t)$ to satisfy:

$$\frac{\partial u}{\partial t} - \Delta u = u^{(1)}(x,t), \;\; u|_{t=0} = \psi_0(x), \tag{2.162}$$

where $u^{(1)}$ satisfies:

$$\frac{\partial u^{(1)}}{\partial t} - \Delta u^{(1)} = u^{(2)}(x,t), \;\; u^{(1)}|_{t=0} = \psi_1(x), \tag{2.163}$$

till recursively

$$\frac{\partial u^{([\frac{l}{2}])}}{\partial t} - \Delta u^{([\frac{l}{2}])} = 0, \;\; u^{([\frac{l}{2}])}|_{t=0} = \psi_{[\frac{l}{2}]}(x), \tag{2.164}$$

in which $\psi_j = \sum_{i=0}^{j} \binom{j}{i} (-1)^i \Delta^i \phi_{j-i}(x)$. Therefore, for $j = 0, .., [\frac{l}{2}]$,

$$(\frac{\partial}{\partial t} - \Delta)^j u(x,t)|_{t=0} = \psi_j(x). \tag{2.165}$$

Then we need to prove that the function u constructed actually satisfies condition (2.160). Notice

$$
\begin{aligned}
&(\frac{\partial}{\partial t} - \Delta)^j u(x,t)|_{t=0} - \psi_j(x) \\
&\sum_{i=0}^{j} \binom{j}{i} (-1)^i \Delta^i [\frac{\partial^{j-i} u(x,t)}{\partial t^{j-i}}|_{t=0} - \phi_{j-i}(x)]
\end{aligned}
\tag{2.166}
$$

thus substitute j = 0 in, by (2.165), we would get $u(x,0) = \phi_0$; then do it inductively, we have (2.160). Therefore, by the properties of the Heat equation,

$$
\begin{aligned}
< u >_{D_{n+1}}^{(l)} &\leq C(< u^{(1)} >_{D_{n+1}}^{(l-2)} + < \psi_0 >_{R_n}^{(l)}) \\
&\leq C(< u^{(2)} >_{D_{n+1}}^{(l-4)} + < \psi_0 >_{R_n}^{(l)} + < \psi_1 >_{R_n}^{(l-2)}) \\
&\leq C \sum_{j=0}^{[\frac{l}{2}]} < \psi_j >_{R_n}^{(l-2j)} .
\end{aligned}
\tag{2.167}
$$

Thus, by definition of $\{\psi_j\}$, (2.161) is proved. For $l < 2$, it is easy to show that $u(x,t) = \phi_0(x)$ satisfies the above condition. Define for simplicity $g(x,t) = \partial_t^r \partial_x^s u, 2r + s < l$. By the equality

$$
\begin{aligned}
g(x,t) = &\sum_{i=0}^{k} \frac{t^i}{i!} \frac{\partial^i g}{\partial \tau^i}|_{t=0} \\
&+ \frac{1}{(k-1)!} \int_0^t (t-\tau)^{k-1} [\frac{\partial^k g(x,\tau)}{\partial \tau^k} - \frac{\partial^k g(x,\tau)}{\partial \tau^k}|_{\tau=0}] d\tau,
\end{aligned}
\tag{2.168}
$$

we get for $\alpha < 1$

$$
|g|_{D_{n+1}^{(\tau)}}^{(0)} \leq C[\sum_{i=0}^{k} max|\frac{\partial^i g}{\partial \tau^i}|_{\tau=0}| + < g >_{t,D_{n+1}^{(\tau)}}^{(k+\alpha)}]. \tag{2.169}
$$

Thus by taking $k = [\frac{l-2r-s}{2}], \alpha = \frac{l-2r-s}{2} - k$, we would get the first term on the right bounded by $\sum_{j=0}^{[\frac{l}{2}]} |\phi_j|_{R_n}^{(0)}$, and the second term is bounded by

$$< \partial_t^r \partial_x^s u >_{t,D_{n+1}^{(\tau)}}^{\frac{l-2r-s}{2}} \leq < u >_{D_{n+1}}^{(l)} \leq \sum_j < \phi_j >_{R_n}^{(l-2j)} . \tag{2.170}$$

by the above proven (2.167). Thus eventually we have $|u|_{D_{n+1}^{(T)}}^{(l)} \leq C \sum_0^{[\frac{l}{2}]} |\phi_j|_{\mathbb{R}_n}^{(l-2j)}$.

Furthermore, we will establish lemmas that will be used later in the proof:

Lemma 2.2.4 *For any $\Omega' \subset \Omega$, and a region Q'_τ, if $u \in \underset{0}{H}^{l,\frac{l}{2}}(Q'_\tau)$, then for any positive non-integral $l > 0$, for integer $j < l$, we have*

$$< u >_{Q'_\tau}^{(j)} \leq C\tau^{\frac{l-j}{2}} < u >_{t,Q'_\tau}^{(\frac{l}{2})} \leq C\tau^{\frac{l-j}{2}} < u >_{Q'_\tau}^{(l)} . \tag{2.171}$$

Proof. The right hand side can be derived directly be the definition of the non-integral part of the Hölder continuous definition (2.130). Also by definition, we know that

$$< u >_{t,Q'_\tau}^{\frac{l}{2}} = \sum_{0<l-2r-s<2} < \partial_t^r \partial_x^s u >_{t,Q'_\tau}^{\frac{l-2r-s}{2}}; \tag{2.172}$$

Therefore, $2r + s = [l], [l] - 1$; for instance if $2r + s = [l]$, we get the right hand side of the above equation equal to

$$\sum_{2r+s=[l]} < \partial_t^r \partial_x^s u >_{t,Q_{\tau'}}^{\frac{l-2r-s}{2}} = \sum_{2r+s=[l]} sup_{x,t,t'} \frac{|\partial_t^r \partial_x^s u(x,t) - \partial_t^r \partial_x^s u(x,t')|}{|t-t'|^{\frac{l-2r-s}{2}}} \tag{2.173}$$

Since $u \in \underset{0}{H}^{l,\frac{l}{2}}(Q_{\tau'})$, we get:

$$\frac{|\partial_t^r \partial_x^s u(x,t) - \partial_t^r \partial_x^s u(x,0)|}{|t-0|^{\frac{l-2r-s}{2}}} = \frac{|\partial_t^r \partial_x^s u(x,t)|}{|t|^{\frac{l-2r-s}{2}}}, \tag{2.174}$$

Therefore, by (2.173), $|\partial_t^r \partial_x^s u(x,t)| \leq Ct^{\frac{l-2r-s}{2}} < u >_{t,Q_{\tau'}}^{\frac{l}{2}} \leq C\tau^{\frac{l-2r-s}{2}} < u >_{t,Q_{\tau'}}^{\frac{l}{2}}$. Thus for any pair (r, s) such that $j = 2r + s = [l]$,

$$< u >^{(j)}_{Q_{\tau'}} \le C\tau^{\frac{l-j}{2}} < u >^{(\frac{l}{2})}_{t,Q_{\tau'}} \tag{2.175}$$

In exactly the same way, the above is true for $j = [l] - 1$. Therefore, if we can prove for $l > 2$,

$$< u >^{[l]-2} \le C\tau < u >^{[l]}, \tag{2.176}$$

then by induction the proof is done. To this end, note that

$$< u >^{[l]-2} = \sum_{2r+s=[l]-2} max|\partial_t^r \partial_x^s u|. \tag{2.177}$$

For $g(x, t) = \partial_t^r \partial_x^s u$, notice that

$$|g(x, t) - g(x, 0)| = |\int_0^t \frac{\partial g}{\partial s} ds| \le |t| max|\frac{\partial g}{\partial t}|; \tag{2.178}$$

Also remember $u \in \underset{0}{H}^{l,\frac{l}{2}}(Q'_\tau)$,

$$|\partial_t^r \partial_x^s u(x, t)| = |\partial_t^r \partial_x^s u(x, t) - \partial_t^r \partial_x^s u(x, 0)| \le C|t| \sum_{2r+s=[l]} max|\partial_t^r \partial_x^s u(x, t)| \le C\tau < u >^{[l]}; \tag{2.179}$$

Therefore, the theorem is proved.

Later on we wish to bound the Hölder norm of θ; in order to do so, we will make use of the small regions defined with the help of Ω^k:

$$Q^k_\tau := \Omega^k \times (0, \tau). \tag{2.180}$$

Define another norm of a function $\theta \in H^{l,\frac{l}{2}}(\overline{Q_T})$ for $T \ge \tau$:

$$\{\theta\}^{(l)}_{Q_\tau} := \sup_k < \theta >^{(l)}_{Q^k_\tau}. \tag{2.181}$$

The following lemma states that the new norm $\{\theta\}^{(l)}_{Q_\tau}$ and $<\theta>^{(l)}_{Q_\tau}$ are equivalent under some suitable conditions:

Lemma 2.2.5 *For $\tau = \lambda^2\kappa_0$, where $\kappa_0 < 1$ denotes a positive constant, then if $\theta \in \underset{0}{H}^{l,\frac{l}{2}(\overline{Q_T})}$, we have*

$$\{\theta\}^{(l)}_{Q_\tau} \leq C <\theta>^{(l)}_{Q_\tau} \leq C\{\theta\}^{(l)}_{Q_\tau}. \tag{2.182}$$

Proof. By the definition above, we only need to prove the right hand side of the inequality. Remember

$$<\theta>^{(l)}_{Q_\tau} = <\theta>^{(l)}_{x,Q_\tau} + <\theta>^{(l/2)}_{t,Q_\tau} \tag{2.183}$$

and clearly for t, the Hölder continuity and its norm does not depend on the division of Ω, thus

$$<\theta>^{(l/2)}_{t,Q_\tau} \leq C\{\theta\}^{(l)}_{Q_\tau}. \tag{2.184}$$

And we only need to show that:

$$<\theta>^{(l)}_{x,Q_\tau} \leq C\{\theta\}^{(l)}_{Q_\tau} \tag{2.185}$$

If for x_0, x_0' satisfying $|x_0 - x_0'| \leq d\lambda$, by the property of the region, they belong to some same Ω^k; therefore, the Hölder norms for points belonging to the same Ω^k in the old and new definition can be evaluated in the same region, and in this case $<\theta>^{(l)}_{x,Q_\tau} \leq 2\{\theta\}^{(l)}_{Q_\tau}$. So we only need to consider the case $|x_0 - x_0'| \geq d\lambda$. For $2r + s = [l]$, we can find x_0, x_0', t such that

$$\frac{|\partial_t^r\partial_x^s\theta|_{x_0,t} - \partial_t^r\partial_x^s\theta|_{x_0',t}|}{|x_0 - x_0'|^{l-[l]}} \geq \frac{1}{2}\underset{x,x',t}{sup}\frac{|\partial_t^r\partial_x^s\theta(x,t) - \partial_t^r\partial_x^s\theta(x',t)|}{|x - x'|^{l-[l]}}; \tag{2.186}$$

since $|x_0 - x_0'| \geq d\lambda$, we get

$$\frac{|\partial_t^r\partial_x^s\theta|_{x_0,t} - \partial_t^r\partial_x^s\theta|_{x_0',t}|}{|x_0 - x_0'|^{l-[l]}} \leq C(\frac{1}{d\lambda})^{l-[l]}sup_k\underset{Q_\tau^k}{sup}|\partial_t^r\partial_x^s\theta|. \tag{2.187}$$

By making use of the previous lemma, and $\tau = \lambda^2 \kappa_0$, we get

$$\begin{aligned} \underset{Q_\tau^k}{sup} |\partial_t^r \partial_x^s \theta| &\leq C \tau^{\frac{l-[l]}{2}} < \theta >_{t,Q_\tau^k}^{(\frac{l}{2}} = C \lambda^{l-[l]} \kappa_0^{\frac{l-[l]}{2}} < \theta >_{t,Q_\tau^k}^{(\frac{l}{2}} \\ &\leq C \lambda^{l-[l]} \kappa_0^{\frac{l-[l]}{2}} < \theta >_{Q_\tau^k}^{l} \end{aligned} \tag{2.188}$$

Therefore, by combining (2.186), (2.187), (2.188), we get $< \theta >_{x,Q_\tau}^{(l)} \leq C \underset{k}{sup} < \theta >_{Q_\tau^k}^{(l)}$, the proof is done.

Since our strategy is to tackle the norm piece-wise, we will establish a lemma that connects the norms of the pieces to the norm of the whole region and function:

Lemma 2.2.6 *If a function $w(x,t) = \sum_k w^k(x,t)$, and $w^k = 0$ for $x \in \Omega \backslash \Omega^k$; then*

$$\{w\}_{Q_\tau}^{(l)} \leq N_0 \underset{k}{sup} < w^k >_{Q_\tau^k}^{(l)} \tag{2.189}$$

Proof. We need only to prove for each k,

$$< w >_{Q_\tau^k}^{(l)} \leq N_0 \underset{k}{sup} < w^k >_{Q_\tau^k}^{(l)}; \tag{2.190}$$

after which by taking sup on the left, the lemma is proved. By definition,

$$< w >_{Q_\tau^k}^{(l)} = < \sum_i w^i >_{Q_\tau^k}^{(l)} \leq \sum_i < w^i >_{Q_\tau^k}^{(l)}, \tag{2.191}$$

for each k fixed, by property 3 of the region Ω, there can be at most N_0 regions indexed by $k_s, s = 1, ..., N_0$ such that Ω^k is covered by them. Therefore

$$\sum_i < w^i >_{Q_\tau^k}^{(l)} = \sum_{s=1}^{N_0} < w^{i_s} >_{Q_\tau^k}^{(l)} \leq \sum_{s=1}^{N_0} < w^{i_s} >_{Q_\tau^{i_s}}^{(l)} \tag{2.192}$$

Then since for each s,

$$< w^{i_s} >_{Q_\tau^{i_s}}^{(l)} \leq \underset{k}{sup} < w^k >_{Q_\tau^k}^{(l)}, \tag{2.193}$$

we then have by (2.192)

$$< w >^{(l)}_{Q^k_\tau} \leq N_0 \sup_k < w^k >^{(l)}_{Q^k_\tau}, \tag{2.194}$$

and the lemma is proved.

2.2.5 Regularity for the zero initial condition

We now first formulate the structure of our proof for the regularity result, given vanishing initial conditions described above in Theorem 2.2.2. For a time τ which will be fixed later, we assign an operator A to map a function θ from the space $\underset{0}{H}^{l+2,\frac{l}{2}+1}(\overline{Q_\tau})$ into a pair of functions $(\mathbb{L}\theta, \mathbb{B}\theta|_{S_\tau})$. We define the Banach space

$$\mathbb{D}^{(l)} = \underset{0}{H}^{l,\frac{l}{2}}(\overline{Q_\tau}) \times \underset{0}{H}^{l+1,\frac{l+1}{2}}(\overline{S_\tau}); \tag{2.195}$$

thus the pair of functions $h = (f, \Phi)$ which has norm $|h|_{\mathbb{D}^{(l)}} := |f|^{(l)}_{Q_\tau} + |\Phi|^{(l+1)}_{S_\tau}$, with conditions described in the theorem 2.2.2 belongs to $\mathbb{D}^{(l)}$. By the character of the coefficients and the Hölder continuous norm, we have

$$|\mathbb{L}\theta|^{(l)}_{Q_\tau} + |\mathbb{B}\theta|^{(l+1)}_{S_\tau} \leq C|\theta|^{(l+2)}_{Q_\tau}. \tag{2.196}$$

In our effort of trying to prove the theorem, essentially we are looking for the valid mapping:

$$A\theta = h, \tag{2.197}$$

where $\theta \in \underset{0}{H}^{l+2,\frac{l}{2}+1}(\overline{Q_\tau})$, and $h \in \mathbb{D}^{(l)}$. (2.196) also implies that A is a bounded operator. To establish such an operator and prove Theorem 2.2.2, it is crucial to find the inverse mapping A^{-1} of A, and prove that A^{-1} is bounded. If we can find such a suitable operator

$A^\star$ that maps from $\mathbb{D}^{(l)}$ to $\underset{0}{H}^{l+2,\frac{l}{2}+1}(\overline{Q_\tau})$, and if we can construct

$$
\begin{aligned}
AA^\star h &= h + Th \\
A^\star A v &= v + Wv
\end{aligned}
\tag{2.198}
$$

for $h \in \mathbb{D}^{(l)}$, $v \in \underset{0}{H}^{l+2,\frac{l}{2}+1}(\overline{Q_\tau})$, then T is an operator in the space of $\mathbb{D}^{(l)}$, and W in $\underset{0}{H}^{l+2,\frac{l}{2}+1}(\overline{Q_\tau})$. Assume if when τ small, and we can prove that

$$
\|T\| \leq 1, \quad \|W\| \leq 1;
\tag{2.199}
$$

therefore, by the contraction mapping principle, $(I+T)^{-1}$ and $(I+W)^{-1}$ exists. Thus from (2.198) we get:

$$
\begin{aligned}
&AA^\star (I+T)^{-1} h = h \\
&(I+W)^{-1} A^\star A v = v.
\end{aligned}
\tag{2.200}
$$

So A has right and left inverse, thus then we can claim A is one to one, and A^{-1} exists, having the property:

$$
\begin{aligned}
\left\|A^{-1}\right\| &\leq \left\|(I+T)^{-1}\right\| \|A^\star\| \\
\left\|A^{-1}\right\| &\leq \left\|(I+W)^{-1}\right\| \|A^\star\|
\end{aligned}
\tag{2.201}
$$

When $\|(I+T)^{-1}\|$ sufficiently small, if further we can demonstrate that $A^\star$ is bounded, then from (2.196) we can further claim that A^{-1} and A are both bounded. Therefore, our main goal in proving the regularity theorem 2.2.2 for zero initial condition would be to construct the operator $A^\star$, and then evaluate the norms of $A^\star$, T, and W.

2.2.6 Construction of the operator $A^\star$

For a fixed small λ, we divide Ω into two subsets $\{\omega^k\}$ and $\{\Omega^k\}$ with conditions described in the section 2.2.3. We denote Z_k as the operator that transforms any function $f(z)$ defined in the local coordinates $\{z\}$ into the function written in the original coordinates $\{x\}$. Also we assign the "principle" part of $\mathbb{L}$ to be

$$\mathbb{L}_0(x,t,\frac{\partial}{\partial x},\frac{\partial}{\partial t}) = \frac{\partial}{\partial t} - \frac{\kappa}{\rho(x,t)}\Delta; \tag{2.202}$$

which will be employed for $k \in R_{in}$. For $k \in R_{on}$, $\mathbb{L}_0^k$ is the operator $\mathbb{L}_0$ in the local coordinate system $\{y\}$ defined around the point ξ^k on the boundary:

$$\mathbb{L}_0^k(y,t,\frac{\partial}{\partial y},\frac{\partial}{\partial t}) = \frac{\partial}{\partial t} - \sum_{r,s} a_{rs}^k \frac{\partial^2}{\partial y_r \partial y_s}; \tag{2.203}$$

here $a_{rs}^k = \sum_{i,j}(\delta^{ij}\frac{\kappa}{\rho})\beta_{ri}^k\beta_{sj}^k$, in which β_{rs}^k is the transformation matrix that maps from $\{x\}$ to $\{y\}$. Similarly, we can define $\mathbb{B}_0 = \mathbb{B}$, and the operator $\mathbb{B}_0^k$ on the surface in terms of coordinate $\{y\}$:

$$\mathbb{B}_0^k(y,t,\frac{\partial}{\partial y}) = \sum_i \mathbf{n}_i^k \frac{\partial}{\partial x}, \tag{2.204}$$

where $\mathbf{n}_i^k = \sum_{j=1}^n \mathbf{n}_j \beta_{ij}^k$. After those definitions, we can use the "freezing coefficient" method to dissect the problem: for $k \in R_{in}$, $f^k \in \underset{0}{H}^{l,\frac{l}{2}}(\overline{D_{n+1}^{(\tau)}})$, we formulate the problem by the operator in the coordinate of $\{x\}$, and with fixed coefficients at point ξ^k:

$$\mathbb{L}_0(\xi^k, 0, \frac{\partial}{\partial x},\frac{\partial}{\partial t})\theta^k(x,t) = f^k(x,t), \tag{2.205}$$

and we are looking for the solution $\theta^k \in \underset{0}{H}^{l+2,\frac{l}{2}+1}(\overline{\mathbb{D}_{n+1}^{(\tau)}})$, and $\mathbb{D}_{n+1}^{(\tau)}$ is defined in the same way as (2.143):

$$\mathbb{D}_{n+1}^{(\tau)} = \mathbb{R}_n \times (0,\tau). \tag{2.206}$$

We can derive from the uniqueness and the existence of the heat equation that the solution for (2.185) exists and is unique. We use A^k to denote the operator that maps f^k to θ^k. For $k \in R_{on}$, i.e., the $\{\Omega^k\}$ that are adjacent to the boundary, for $f^k(z,t) \in \underset{0}{H}^{l,\frac{l}{2}}(\overline{R^{(\tau)}}), \Phi^k(z',t) \in \underset{0}{H}^{l+1,\frac{l+1}{2}}(\overline{\tilde{D_n}^{(\tau)}})$, we name $h^k = (f^k, \Phi^k)$, and thus the boundary value problem with fixed coefficient defined at the local coordinate $\{z\}$:

$$\begin{aligned} &\mathbb{L}_0^k(\xi^k, 0, \frac{\partial}{\partial z}, \frac{\partial}{\partial t}) = f^k(z,t) \\ &\mathbb{B}_0^k(\xi^k, 0, \frac{\partial}{\partial z})\theta^k|_{z_n=0} = \Phi^k(z',t) \end{aligned} \tag{2.207}$$

can also be shown to have a unique solution in the class of $\underset{0}{H}^{l+2,\frac{l}{2}+1}(\overline{R^{(\tau)}})$. In this scenario A^k is the operator that maps from h^k to θ^k. From the known theorems of the fixed coefficient,

$$\begin{aligned} &< A^k f^k >^{(l+2)}_{\mathbb{D}^{(\tau)}_{n+1}} \leq C < f^k >^{(l)}_{\mathbb{D}^{(\tau)}_{n+1}}, \text{for k} \in R_{in}; \\ &< A^k(f^k, \Phi^k) >^{(l+2)_{R^{(\tau)}}} \leq C(< f^k >^{(l)}_{R^{(\tau)}} + < \Phi^k >^{(l+2)}_{\tilde{D_n}^{(\tau)}}), \text{for k} \in R_{on}. \end{aligned} \tag{2.208}$$

Then we can define:

$$A^\star h = \sum_k \eta^k(x)\theta^k(x,t), \tag{2.209}$$

where $h = (f, \Phi)$, and θ^k is defined as:

$$\theta^k(x,t) = \begin{cases} A^k \zeta^k f, \text{for } k \in R_{in} \\ Z_k A^k(Z_k K^{-1}\zeta^k f, {Z_K}^{-1}\zeta^k\Phi), \text{for } k \in R_{on}; \end{cases} \tag{2.210}$$

Thus we need to prove $A^\star$ is bounded. Notice for $k \in R_{on}$, $Z_k, {Z_K}^{-1}$ transform between different coordinate system for functions to be properly defined. By Lemma 2.2.5, we have

$$\{A^\star h\} \leq C \underset{k}{sup} < \theta^k >^{(l+2)}_{Q^k_\tau} \tag{2.211}$$

By (2.208),

$$
\begin{aligned}
&< \theta^k >^{(l+2)}_{Q^k_\tau} \leq C < \zeta^k f >^{(l)}_{Q^k_\tau}, \text{ for } k \in R_{in} \\
&< \theta^k >^{(l+2)}_{Q^k_\tau} \leq C(< \zeta^k f >^{(l)}_{Q^k_\tau} + < {Z_K}^{-1} \zeta^k \Phi >^{(l+1)}_{Q^k_\tau}), \text{ for } k \in R_{on}
\end{aligned}
\tag{2.212}
$$

And the right hand side expression of the second inequality in (2.212) is the norm for h in $\mathbb{D}^{(l)}$. Therefore combined the above results, while remembering that the norm $|\cdot|$ is equivalent to $\{\cdot\}$, we have

$$
\{A^\star h\}^{(l+2)}_{Q_\tau} \leq C \left\|h\right\|_{\mathbb{D}^{(l)}} . \tag{2.213}
$$

Thus $A^\star$ is bounded. Now we need to establish (2.200), and then to evaluate the bounds for T and W. We define $\mathbb{L}_1$, which contains the lower order derivatives for $\mathbb{L}$:

$$
\mathbb{L}_1(x, t, \frac{\partial}{\partial x})\theta = \mathbf{u} \cdot \nabla \theta. \tag{2.214}
$$

We write $\mathbb{L}_0 A^\star h$ in the following way:

$$
\begin{aligned}
\mathbb{L}_0 A^\star h = \sum_k \mathbb{L}_0 \eta^k \theta^k = &\sum_k (\mathbb{L}_0(x, t, \frac{\partial}{\partial x}, \frac{\partial}{\partial t})\eta^k\theta^k - \eta^k \mathbb{L}_0(x, t, \frac{\partial}{\partial x}, \frac{\partial}{\partial t})\theta^k) \\
&+ \sum_k \eta^k [\mathbb{L}_0(x, t, \frac{\partial}{\partial x}, \frac{\partial}{\partial t}) - \mathbb{L}_0(\xi^k, 0, \frac{\partial}{\partial x}, \frac{\partial}{\partial t})]\theta^k \\
&+ \sum_k \eta^k \mathbb{L}_0(\xi^k, 0, \frac{\partial}{\partial x}, \frac{\partial}{\partial t})\theta^k.
\end{aligned}
\tag{2.215}
$$

By definition of A^k, for $k \in R_{in}$, $\mathbb{L}_0(x, t, \frac{\partial}{\partial x}, \frac{\partial}{\partial t})$ and A^k are inverse operations to each other, thus

$$
\mathbb{L}_0(\xi^k, 0, \frac{\partial}{\partial x}, \frac{\partial}{\partial t})\theta^k = \mathbb{L}_0(\xi^k, 0, \frac{\partial}{\partial x}, \frac{\partial}{\partial t})A^k \zeta^k f = \zeta^k f; \tag{2.216}
$$

whereas for $k \in R_{on}$,

$$
\mathbb{L}^k_0(\xi^k, 0, \frac{\partial}{\partial z}, \frac{\partial}{\partial t})A^k({Z_K}^{-1}\zeta^k f, {Z_K}^{-1}\zeta^k \Phi) = {Z_K}^{-1}\zeta^k f. \tag{2.217}
$$

Under the change of variables (2.146), we get

$$\frac{\partial}{\partial y} = \frac{\partial}{\partial z} - \frac{\partial}{\partial z_n}\nabla F, \tag{2.218}$$

where $\nabla F = (\frac{\partial}{\partial z_1}, ..., \frac{\partial}{\partial z_{n-1}}, 0)$. Therefore, for $k \in R_{on}$, we use the $Z_k, {Z_K}^{-1}$ to transform coordinate systems to get:

$$\begin{aligned}
&\mathbb{L}_0(\xi^k, 0, \frac{\partial}{\partial x}, \frac{\partial}{\partial t})\theta^k \\
&= \mathbb{L}_0(\xi^k, 0, \frac{\partial}{\partial x}, \frac{\partial}{\partial t}) Z_k A^k({Z_K}^{-1}\zeta^k f, {Z_K}^{-1}\zeta^k \Phi) \\
&= Z_k[\mathbb{L}_0^k(\xi^k, 0, \frac{\partial}{\partial z} - \nabla F \frac{\partial}{\partial z_n}, \frac{\partial}{\partial t})] A^k({Z_K}^{-1}\zeta^k f, {Z_K}^{-1}\zeta^k \Phi).
\end{aligned} \tag{2.219}$$

Also for $k \in R_{on}$, by (2.217) and the definition of $A^\star$,

$$\sum_{k\in R_{on}} \eta^k Z_k[\mathbb{L}_0^k(\xi^k, 0, \frac{\partial}{\partial z}, \frac{\partial}{\partial t}) A^k({Z_K}^{-1}\zeta^k f, {Z_K}^{-1}\zeta^k \Phi)] = \sum_{k\in R_{on}} \eta^k \zeta^k f. \tag{2.220}$$

Combine the above results,

$$\begin{aligned}
\mathbb{L}A^\star h &= \mathbb{L}_0 A^\star h + \mathbb{L}_1 A^\star h \\
&= f + T_1 h,
\end{aligned} \tag{2.221}$$

where T_1 can be evaluated into four following terms:

$$\begin{aligned}
T_1 h &= \mathbb{L}_1 A^\star h + \sum_k (\mathbb{L}_0(x, t, \frac{\partial}{\partial x}, \frac{\partial}{\partial t})\eta^k\theta^k - \eta^k \mathbb{L}_0(x, t, \frac{\partial}{\partial x}, \frac{\partial}{\partial t})\theta^k) \\
&+ \sum_k \eta^k[\mathbb{L}_0(x, t, \frac{\partial}{\partial x}, \frac{\partial}{\partial t}) - \mathbb{L}_0(\xi^k, 0, \frac{\partial}{\partial x}, \frac{\partial}{\partial t})]\theta^k \\
&+ \sum_{k\in R_{on}} \eta^k Z_k[\mathbb{L}_0^k(\xi^k, 0, \frac{\partial}{\partial z} - \frac{\partial}{\partial z_n}\nabla F, \frac{\partial}{\partial t}) - \mathbb{L}_0^k(\xi^k, 0, \frac{\partial}{\partial z}, \frac{\partial}{\partial t})] A^k({Z_K}^{-1}\zeta^k f, {Z_K}^{-1}\zeta^k \Phi) \\
&:= I_1 + I_2 + I_3 + I_4.
\end{aligned} \tag{2.222}$$

Also, similarly we have

$$\mathbb{B}A^{\star}h|_{S_\tau} = \Phi + T_2 h \tag{2.223}$$

where the T_2h can be expressed as:

$$\begin{aligned} T_2 h = &\sum_{k\in R_{on}} \{(\mathbb{B}_0\eta^k\theta^k - \eta^k\mathbb{B}_0\theta^k) + \eta^k[\mathbb{B}_0(x,t) - \mathbb{B}_0(\xi^k,0)]\}|_{s_\tau} \\ &+ \eta^k Z_K[\mathbb{B}_0^k(\xi^k, 0, \frac{\partial}{\partial z} - \frac{\partial}{\partial z_n}\nabla F) - \mathbb{B}_0^k(\xi^k, 0, \frac{\partial}{\partial z})]A^k({Z_K}^{-1}\zeta^k f, {Z_K}^{-1}\zeta^k\Phi). \end{aligned} \tag{2.224}$$

With the above notations in (2.201) and (2.204),

$$AA^{\star}h = (f,\Phi) + (T_1h, T_2h) = h + Th. \tag{2.225}$$

Thus in order to bound T, we need to bound T_1 and T_2, and the calculations are carried out as follows: For I_1 in (2.202),

$$\{\mathbb{L}_1 A^{\star}h\}^{(l)}_{Q_\tau} = \{(\mathbf{u}\cdot\nabla)A^{\star}h\}^{(l)}_{Q_\tau} \le CA^{\star}h^{(l+1)}_{Q_\tau}; \tag{2.226}$$

By Lemma 2.2.3 and Lemma 2.2.5, and by the characteristic of function η^k, we have

$$\{I_2\}^{(l)}_{Q_\tau} \le C(\frac{1}{\lambda}\sup_k <\theta^k>^{(l+1)}_{Q^k_\tau} + \frac{1}{\lambda^2}\sup_k <\theta^k>^{(l)}_{Q^k_\tau}); \tag{2.227}$$

For I_3, since η^k bounded,

$$\{I_3\}^{(l)}_{Q_\tau} \le C\sum_k\sum_i |[\frac{\kappa}{\rho(x,t)} - \frac{\kappa}{\rho(\xi^k,0)}]\frac{\partial^2\theta^k}{\partial x_i^2}|; \tag{2.228}$$

We denote $g(x,t) := \frac{\kappa}{\rho(x,t)} \in \underset{0}{H}^{l,\frac{l}{2}}(\overline{Q_\tau})$; and also

$$|g(x,t) - g(\xi^k,0)| \le |g(x,t) - g(\xi^k,t)| + |g(\xi^k,t) - g(\xi^k,0)|; \tag{2.229}$$

for $x \in \Omega^k, l < 1$,

$$\frac{|g(x,t) - g(\xi^k,t)|}{|x - \xi^k|^l} \leq |g|^{(l)}_{\Omega^k}, \tag{2.230}$$

which means $|g(x,t) - g(\xi^k,t)| \leq C\lambda^l$, since the distance between points in the same region Ω^k can also be measured by λ. Also by Lemma 2.2.4, $\tau = \lambda^2\kappa_0$ for some constant κ_0, we can see that

$$\begin{aligned} |g(\xi^k,t) - g(\xi^k,0)| &\leq C|t-0|^{(\frac{l}{2})} \\ &\leq C\tau^{\frac{l}{2}} \leq C\lambda^l; \end{aligned} \tag{2.231}$$

thus in sum we have $|g(x,t) - g(\xi^k,0)| \leq C\lambda^l$. Thus by (2.228),

$$\{I_3\}^{(l)}_{Q_\tau} \leq C\lambda^l \underset{k}{sup} < \theta^k >^{(l+2)}_{Q^k_\tau} . \tag{2.232}$$

Notice given the transformation, when the point is on the surface $S \in H^{(l+2)}$, $|\frac{\partial F}{\partial Z_i}| \leq C\lambda$, thus similar to I_1, I_2, for $\lambda < 1$,

$$\{I_4\}^{(l)}_{Q_\tau} \leq \lambda \underset{k}{sup} < \theta^k >^{(l+2)}_{Q^k_\tau} . \tag{2.233}$$

Thus

$$\begin{aligned} \{T_1 h\}^{(l)}_{Q_\tau} &\leq C(\tau^{\frac{1}{2}}\{A^\star h\}^{(l+2)}_{Q_\tau} + \frac{1}{\lambda}\underset{k}{sup} < \theta^k >^{(l+1)}_{Q^k_\tau} \\ &+ \frac{1}{\lambda^2}\underset{k}{sup} < \theta^k >^{(l)}_{Q^k_\tau} + \lambda^l \underset{k}{sup} < \theta^k >^{(l+2)}_{Q^k_\tau}) \\ &\leq C(\tau^{\frac{1}{2}}\{A^\star h\}^{(l+2)}_{Q_\tau} + \frac{1}{\lambda}\tau^{\frac{1}{2}}\underset{k}{sup} < \theta^k >^{(l+2)}_{Q^k_\tau} \\ &+ \frac{1}{\lambda^2}\tau \underset{k}{sup} < \theta^k >^{(l+2)}_{Q^k_\tau} + \lambda^l \underset{k}{sup} < \theta^k >^{(l+2)}_{Q^k_\tau}) \end{aligned} \tag{2.234}$$

Therefore by (2.212) and (2.213), we have altogether

$$\{T_1 h\}^{(l)}_{Q_\tau} \leq C(\lambda\kappa_0^{\frac{1}{2}} + \kappa_0^{\frac{1}{2}} + \kappa_0 + \lambda^l) \left\| h \right\|_{\mathbb{D}^{(l)}} . \tag{2.235}$$

for κ_0, λ small, the norm of T_1 will be bounded by 1; also in the same manner, we can demonstrate the norm of T_2 small, therefore the norm of T small. Analogously, we can construct $A^\star Av = v + Wv$, and the norm of W can be bounded by 1. Therefore, combined with the general structure that is already in place, the existence in Hölder class of the regularity theorem given zero initial condition is proved. Also since we have proven A^{-1} bounded, then the inequality of Theorem 2.2.2 is established.

2.2.7 Proof of the general regularity theorem

Now we aim to loosen the zero initial condition. Recall by $\theta_{(k)}$, we mean the kth-order compatibility condition, for $k = 0, ..., [\frac{l}{2}] + 1$. Therefore by (2.138), we have:

$$\begin{aligned} |\theta_{(0)}(x)|^{(l+2)} &= |\theta_0|^{l+2} \\ |\theta_{(0)}(x)|^{(l+2-2)} &= |f(x,0) + \frac{\kappa}{\rho}\Delta\theta(x,0) - \mathbf{u}(x,0)\nabla\theta(x,0)|^{(l)} \\ &\leq |f|^l + C|\theta_0|_\Omega^{(l+2)}. \end{aligned} \tag{2.236}$$

By the standard extension theorem in Hölder continuous space, we can extend $\theta_{(k)}(x)$ onto $\mathbb{R}_n$ and the norm will be preserved

$$|\theta_{(k)}(x)|_{\mathbb{R}_n}^{(l+2-2k)} \leq C|\theta_{(k)}(x)|_\Omega^{(l+2-2k)}; \tag{2.237}$$

here $k = 0, 1$. Then by theorem 2.2.3, we can construct a function $\tilde{\theta}(x,t) \in H^{l+2,\frac{l}{2}+1}(\overline{D_{n+1}^{(\tau)}})$, satisfying:

$$\frac{\partial^k \tilde{\theta}}{\partial t^k}|_{t=0} = \theta_{(k)}(x), \text{ for } k = 0, 1, \tag{2.238}$$

and by (2.236) and Theorem 2.2.3,

$$\begin{aligned} |\tilde{\theta}|_{D_{n+1}^{(\tau)}}^{(l+2)} &\leq C(|\theta_{(0)}|_{R^n}^{(l+2)} + |\theta_{(1)}|_{R^n}^{(l)}) \\ &\leq (|f|_{Q_\tau}^{(l)} + |\theta_0|_\Omega^{(l+2)}). \end{aligned} \tag{2.239}$$

Therefore, if we define:

$$f' = f - \mathbb{L}\tilde{\theta},\ \Phi' = \Phi - \mathbb{B}\tilde{\theta}, \tag{2.240}$$

both f', ϕ' satisfy the zero initial condition for $l = \frac{1}{6}$. Therefore for the problem

$$\begin{aligned} &\mathbb{L}\theta' = f' \\ &\mathbb{B}\theta'|_{s_\tau} = \Phi' \end{aligned} \tag{2.241}$$

the condition for the zero initial condition as well as the compatibility condition are both satisfied; thus by theorem 2.2.2, the solution $\theta' = \theta - \tilde{\theta}$ exists, and accordingly the solution for Theorem 2.2.1 exists in time $t \in (0, \tau)$, with

$$|\theta|_{Q_\tau}^{(l+2)} \leq |\theta'|_{Q_\tau}^{(l+2)} + |\tilde{\theta}|_{Q_\tau}^{(l+2)} \leq C(|f|_{Q_\tau}^{(l)} + |\theta_0|_{\Omega}^{(l+2)} + |\Phi|_{S_\tau}^{(l+1)}). \tag{2.242}$$

To extend the region in time, we only need to choose the new start time $T_0 = \frac{\tau}{2}$, and apply the above procedure while considering

$$\begin{aligned} &\theta(x,t)|_{t=\frac{\tau}{2}} := \theta_{(\frac{\tau}{2},0)}(x); \\ &\frac{\partial\theta}{\partial t}|_{\frac{\tau}{2}} := \theta_{(\frac{\tau}{2},1)}(x); \end{aligned} \tag{2.243}$$

as the new definitions of the initial condition for the new $\tilde{\theta}$ to be compatible with. Thus the proof is complete.

2.2.8 Positivity for the temperature

After proving the existence of the solution for the energy equation in the certain continuous classes, we can prove that the temperature is positive, as long as the solution exists. Remember that we require

$$\theta_0 \geq C_0 > 0 \tag{2.244}$$

for some constant C_0. Since due to Theorem 2.2.1, we can derive the solution $\theta(x,t)$ to fit into the class of at least $C(\overline{Q_T})$; therefore, we can find a certain time t_0, such that for $t \in [0, t_0]$,

$$\theta(x,t) \geq C_1 > 0 \tag{2.245}$$

for some constant C_1. Now we prove the following theorem regarding the positivity of temperature:

Theorem 2.2.7 *Given the solution θ for the system in Theorem 2.2.1 with*

$$\Phi_{S_T} = 0; \tag{2.246}$$

when the initial data satisfies (2.244), then $\theta(x,t) \geq C_0 > 0$ for any $t \in [0,T]$.

Proof. We can rewrite the energy equation in the following way: multiply both sides of the energy equation by an exponential factor $\theta^{-\lambda}$(assume $\lambda > 2$), we get:

$$\rho\theta_t\theta^{-\lambda} + \rho\mathbf{u}\cdot\nabla\theta\theta^{-\lambda} = \kappa\theta^{-\lambda}\Delta\theta + \frac{\mu}{2}\theta^{-\lambda}|\nabla\mathbf{u} + \nabla\mathbf{u}^T|^2 \tag{2.247}$$

By the continuity equation, this form can also be translated into:

$$-\frac{1}{\lambda-1}[\rho\theta^{-(\lambda-1)}]_t - \frac{1}{\lambda-1}div(\rho\mathbf{u}\theta^{-(\lambda-1)}) = \kappa\theta^{-\lambda}\Delta\theta + \frac{\mu}{2}\theta^{-\lambda}|\nabla\mathbf{u} + \nabla\mathbf{u}^T|^2 \tag{2.248}$$

Do integration with respect to the spatial variables, recall that $\mathbf{u}$ vanishes on the boundary and the Neumann Boundary Condition, we would get

$$\begin{aligned} &\frac{d}{dt} - \frac{1}{\lambda-1}\int_\Omega \rho\theta^{-(\lambda-1)}dx = \\ &\lambda\kappa\int_\Omega \theta^{-\lambda-1}|\nabla\theta|^2dx + \frac{\mu}{2}\int_\Omega \theta^{-\lambda}|\nabla\mathbf{u} + \nabla\mathbf{u}^T|^2dx; \end{aligned} \tag{2.249}$$

Therefore we have $\frac{d}{dt}\frac{1}{\lambda-1}\int_\Omega \rho\theta^{-(\lambda-1)}dx \leq 0$, which implies for the time t_0 prescribed above in (2.245), we would have:

$$\frac{1}{\lambda-1}\int_\Omega \rho\theta_0^{-(\lambda-1)}dx \geq \frac{1}{\lambda-1}\int_\Omega \rho\theta(x,t_0)^{-(\lambda-1)}dx \tag{2.250}$$

Since in Q_{t_0}, $0 < min(\rho) \leq \rho \leq max(\rho) < \infty$, we get:

$$max(\rho)\left\|\frac{1}{\theta_0}\right\|_{L^{\lambda-1}}^{\lambda-1} \geq min(\rho)\left\|\frac{1}{\theta(t_0)}\right\|_{L^{\lambda-1}}^{\lambda-1} \tag{2.251}$$

which is the same as

$$max(\rho)^{\frac{1}{\lambda-1}}\left\|\frac{1}{\theta_0}\right\|_{L^{\lambda-1}} \geq min(\rho)^{\frac{1}{\lambda-1}}\left\|\frac{1}{\theta(t_0)}\right\|_{L^{\lambda-1}} \tag{2.252}$$

When λ approaches ∞, we would get

$$\left\|\frac{1}{\theta_0}\right\|_{L^\infty} \geq \left\|\frac{1}{\theta(t_0)}\right\|_{L^\infty}; \tag{2.253}$$

therefore, for any $x \in \Omega$, $\frac{1}{\theta(x,t_0)} \leq \left\|\frac{1}{\theta(t_0)}\right\|_{L^\infty} \leq \frac{1}{C_0}$; thus we get $\theta(x,t_0) \geq C_0$. To extend this proof to the time $t \in [0,T]$, we only need to notice that our pick of t_0 is random as long as $min(\theta(t_0)) > 0$, since we can assign $C_0 := \underset{t=0}{min}(\theta(t))$, then actually we have proven that the minimum of temperature is non-decreasing with respect to time. Thus the whole theorem is proved.

2.2.9 A few more remarks

Before presenting our theorem relating to the temperature, Two remarks are useful.

1. The initial density of the original, unperturbed equation can be represented by:

$$\rho(x,0) = \overline{\rho}(x) + \sigma(x,0) = \overline{\rho}(x) + \sigma_0. \tag{2.254}$$

Since we assumed $\overline{\rho}(x)$ is smooth and positively bounded from below, when we scale the σ_0 by a factor of δ: $\sigma_0 = \delta\overline{\sigma}_0$, the term σ_0 can always be absorbed by for example a half of $\overline{\rho}(x)$. Thus $\rho_0(x)$ can be bounded by $\underset{x\in\Omega}{inf}\overline{\rho}(x)$ and $\underset{x\in\Omega}{sup}\overline{\rho}(x)$, and furthermore, by the divergence free condition, for any time t, density can be bounded by:

$$0 < \frac{1}{2}\underset{x\in\Omega}{inf}\{\overline{\rho}(x)\} \leq \rho(x,t) \leq 2\underset{x\in\Omega}{sup}\{\overline{\rho}(x)\} < \infty. \tag{2.255}$$

2. Previously the nonlinear instability in Theorem 2.1.1 is presented as given any sufficient small ϵ fixed, for any small δ, there exists an escape time T^δ such that the L^2 norm of the density and the velocity will exceed ϵ. Here we modify the statement to show the exponential growth of the two variables for any $t \in (0, T^\delta)$:

Lemma 2.2.8 *For m_0, C_1 the constants presented in theorem 2.1.1, T^δ defined in (2.18), for any time $t \in (0, T^\delta)$, there exists a constant C, which does not depend on t or δ, such that:*

$$\|\sigma(t)\|_{L^2},\ \ \|\boldsymbol{u}\|_{L^2} \geq C\delta e^{\Lambda t}. \tag{2.256}$$

Proof. In the proof of theorem 2.1.1, the L^2 norm the density of the linearized part σ^a satisfies:

$$\left\|\sigma_l^\delta(t)\right\|_{L^2} = \delta e^{\Lambda t}\left\|\overline{\sigma}_0\right\|_{L^2}, \tag{2.257}$$

in which $\overline{\sigma}_0$ is the initial density without the scaling. Therefore, by (2.35) and (2.37):

$$\begin{aligned}\left\|\sigma^\delta(t)\right\|_{L^2} \geq& \left\|\sigma_l^\delta(t)\right\|_{L^2} - \left\|\sigma^d(t)\right\|_{L^2} \\ \geq& \delta e^{\Lambda t}\left\|\overline{\sigma}_0\right\|_{L^2} - \sqrt{C_1}\delta^{\frac{3}{2}}e^{\frac{3\Lambda t}{2}}\end{aligned} \tag{2.258}$$

Denote $2\epsilon_t := \delta e^{\Lambda t}$, $t \in [0, T^\delta]$. Then right hand side of (2.103) is equal to

$$2\epsilon_t\left\|\overline{\sigma}_0\right\|_{L^2} - \sqrt{C_1}\cdot 2^{\frac{3}{2}}\epsilon_t^{\frac{3}{2}} = \epsilon_t\left\|\overline{\sigma}_0\right\|_{L^2} + \epsilon_t\left\|\overline{\sigma}_0\right\|_{L^2} - \sqrt{8C_1}\epsilon_t^{\frac{3}{2}}. \tag{2.259}$$

Since $t \leq T^\delta$, $\epsilon_t = \frac{1}{2}\delta e^{\Lambda t} \leq \frac{1}{2}\delta e^{\Lambda T^\delta} = \epsilon_0$; combined with (2.38), we get

$$\epsilon_t^{\frac{1}{2}} \leq \frac{m_0}{\sqrt{8C_1}}. \tag{2.260}$$

Multiply $\sqrt{8C_1}\epsilon_t$ on both sides to get

$$\sqrt{8C_5}\epsilon_t^{\frac{3}{2}} \leq \epsilon_t m_0 \leq \epsilon_t \left\|\overline{\sigma}_0\right\|_{L^2} \tag{2.261}$$

Therefore, in view of (2.103),(2.106),

$$\begin{aligned} \left\|\sigma^\delta(t)\right\|_{L^2} \geq & \epsilon_t \left\|\overline{\sigma}_0\right\|_{L^2} \\ \geq & C\delta e^{\Lambda t} \end{aligned} \tag{2.262}$$

for some constant C. Similar procedures will yield the growth rate for the velocity **u**, thus the proof is complete.

2.3 Estimates for the energy equation

We now present our theorem in regards to the L^1 norm of the temperature. We are in the position to state our main theorem.

Theorem 2.3.1 *Assume that the RT density profile $\overline{\rho}$ satisfies (2.15). Then the L^1 norm of the temperature is unstable in the Hadamard sense: for positive constant Λ in (2.19), there exists a positive constant $\tilde{\epsilon}$, such that for any positive δ smaller than a constant $\tilde{\delta}_0$, there exists an escape time $t^\delta \in (0, T^{max})$ such that*

$$\sup_{0 \leq t \leq t^\delta} \left\{\left\|\theta(t)\right\|_{L^1}\right\} \geq \tilde{\epsilon}. \tag{2.263}$$

Proof. Integrate (2.11), by Poincare's inequality and the Lemma above:

$$
\begin{aligned}
\int_{\Omega}\rho(x,t)\theta(x,t)d\mathbf{x} &= \mu\int_0^t\int_{\Omega}|\nabla\mathbf{u}(s)|^2 d\mathbf{x}ds + \int_{\Omega}\rho(\mathbf{x},0)\theta(\mathbf{x},0)d\mathbf{x} \\
&\geq C\int_0^t\int_{\Omega}|\mathbf{u}(s)|^2 d\mathbf{x}ds + \int_{\Omega}\rho(\mathbf{x},0)\theta(\mathbf{x},0) \\
&\geq C\int_0^t \delta^2 e^{2\Lambda s}ds + \int_{\Omega}\rho(\mathbf{x},0)\theta(\mathbf{x},0)d\mathbf{x} \\
&\geq C\int_0^t \delta^2 e^{2\Lambda s}ds
\end{aligned}
\tag{2.264}
$$

The second term on the right is dropped since the initial temperature $\theta > 0$. Then we get:

$$\int_{\partial\Omega}\rho(x,t)\theta(x,t)d\mathbf{x} \geq C\delta^2(e^{2\Lambda t}-1) \tag{2.265}$$

when $\delta < \frac{2\epsilon}{m_0}e^{-\frac{1}{2}}$, we get the corresponding escape time t^δ linked by (2.18), and also we have:

$$e^{\Lambda t^\delta} = \frac{2\epsilon}{m_0\delta} \tag{2.266}$$

therefore, $e^{\Lambda t^\delta} > e^{\frac{1}{2}}$, which implies:

$$e^{2\Lambda t^\delta} - 1 > \frac{e^{2\Lambda t^\delta}}{2}. \tag{2.267}$$

Thus for any $\delta < \tilde{\delta_0}$, where $\tilde{\delta_0} = min(\frac{2\epsilon}{m_0}e^{-\frac{1}{2}}, \delta_0)$, with δ_0 defined in Theorem 2.1.1, in view of (2.265),

$$
\begin{aligned}
\int_{\Omega}\rho(x,t^\delta)\theta(x,t^\delta)d\mathbf{x} \geq& \frac{C}{2}\delta^2 e^{2\Lambda t^\delta} \\
\geq& C\hat{\epsilon}^2
\end{aligned}
\tag{2.268}
$$

Where we denote $\hat{\epsilon} = \delta e^{\Lambda t^\delta}$. By the uniform boundedness of ρ described in (2.100), and the positivity of θ, we get:

$$\begin{aligned}\int_\Omega |\theta| d\mathbf{x} &= \int_\Omega \theta d\mathbf{x} \\ &\geq \frac{1}{2\underset{x\in\Omega}{sup}\{\overline{\rho}(x)\}} \int_\Omega \rho\theta d\mathbf{x} \\ &\geq C\hat{\epsilon}^2 := \tilde{\epsilon}.\end{aligned} \tag{2.269}$$

2.3.1 Additional results

Since the region is bounded, by Hölder's inequality we naturally get:

Corollary 2.3.1.1 *Given the same assumptions as in theorem 2.3.1, for positive Λ, there exists a positive constant $\tilde{\epsilon}_0$, such that for any $\delta \in (0, \delta_0)$, there exists a $t^\delta \in (0, T^{max})$, such that*

$$\underset{0\leq t\leq t^\delta}{sup} \{\|\theta(t)\|_{L^2}\} \geq \tilde{\epsilon}_0. \tag{2.270}$$

When δ is chosen to be sufficient small and the theorem 2.3.1 holds, we can observe that the L_1, L_2 norm of θ has a faster growth rate $e^{2\Lambda t}$, when comparing to the growth rate $e^{\Lambda t}$ for the perturbed density and velocity. Also note that in an extreme case we can formulate:

Corollary 2.3.1.2 *Given any small $\epsilon > 0$, then if :*

$$\|\theta_0\|_{L^1} = \epsilon; \tag{2.271}$$

then we can still find a positive $\tilde{\epsilon}$ that satisfies Theorem 2.3.1. In fact, the same $\tilde{\epsilon}$ in Theorem 2.3.1 can also be used for any small initial data condition.

We only need to note that initial condition term of the right hand side in (2.264) is dropped since both $\theta(x,0), \rho(x,0)$ are non-negative. Then for the lower bound which yields instability, the exponential growth only needs to come from the L_2 norm of the gradient of the velocity for each time t, as shown in the lemma 2.2.8, after which we do integration. Since a lower bound of the growth rate of L^1 norm of the temperature has been developed, it is natural to ask if the growth rate is sharp:

Corollary 2.3.1.3 *Assume conditions in theorem 2.3.1 are satisfied, and t^δ is the same as in theorem 2.2.1. Then*

$$\left\|\theta(t^\delta)\right\|_{L^1} \leq C_{(\theta,\rho_0,\boldsymbol{u}_0)} + C_g\delta^2 e^{2\Lambda t^\delta}, \tag{2.272}$$

where $C_{(\theta,\rho_0,\boldsymbol{u}_0)} = \int_\Omega \rho_0\theta_0 + \frac{1}{2}\rho_0|u_0|^2$, and C_g is a constant that depends on g.

Proof. First we multiply **u** for the perturbed momentum equation (2.12) to derive

$$\frac{d}{dt}\int \rho|u|^2 d\mathbf{x} + 2\mu\int |\nabla\mathbf{u}|^2 d\mathbf{x} = -2g\int \sigma u_3 d\mathbf{x} \tag{2.273}$$

where u_3 denotes the third component of the perturbed velocity. Integrate in time to find:

$$\int \rho(t)|\mathbf{u}|^2(t)d\mathbf{x} - \int \rho(0)|\mathbf{u}|^2(0)d\mathbf{x} + 2\mu\int_0^t\int |\nabla\mathbf{u}|^2 d\mathbf{x}ds = -2g\int_0^t\int \sigma u_3 d\mathbf{x}ds \tag{2.274}$$

By (2.264) we have (notice the last term can be dropped on the right side of the equation),

$$\begin{aligned}\int_\Omega \rho(x,t)\theta(x,t)d\mathbf{x} &= \mu\int_0^t\int_\Omega |\nabla\mathbf{u}(s)|^2 d\mathbf{x}ds + \int_\Omega \rho(\mathbf{x},0)\theta(\mathbf{x},0)d\mathbf{x}\\ &= \int_\Omega \rho_0\theta_0 dx - g\int_0^t\int \sigma u_3 d\mathbf{x}ds + \frac{1}{2}\int \rho_0|\mathbf{u}_0|^2 dx - \frac{1}{2}\int \rho(t)|u|^2(t)dx\\ &\leq g\int_0^t\int |\sigma u_3| d\mathbf{x}ds + C_{(\theta,\rho_0,\mathbf{u}_0)}\end{aligned} \tag{2.275}$$

Since $e^{\Lambda t^\delta}$ is the sharp growth rate for both σ and $\mathbf{u}$, by taking $t = t^\delta$, the right hand side can be bounded by:

$$
\begin{aligned}
& g \int_0^{t^\delta} \|\sigma(s)\|_{L^2(\Omega)} \|\mathbf{u}(s)\|_{L^2(\Omega)} \, ds + C_{(\theta,\rho_0,\mathbf{u}_0)} \\
& \leq C_g \delta^2 (e^{2\Lambda t^\delta} - 1) + C_{(\theta,\rho_0,\mathbf{u}_0)} \\
& \leq C_g \delta^2 e^{2\Lambda t^\delta} + C_{(\theta,\rho_0,\mathbf{u}_0)}.
\end{aligned}
\tag{2.276}
$$

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